Lecture 7

- OHs T 2:30-3:30pm, Th 1-2pm
- Makeup lecture this Friday 3:30-4:45pm

Thm: If a_n is bounded and $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n b_n = 0$.

$\boxed{\text{14 Further Limit Theorems}}$

$<$

Thm: Suppose a_n, b_n are convergent sequences with $a_n \le b_n$ for all but finitely many $n \in \mathbb{N}$. Then $\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$.

$<$?

HW4

<u>Thm</u> (Squeeze): Suppose
$$a_n \le b_n \le c_n$$
for all but finitely many $n \in \mathbb{N}$
and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L \in \mathbb{R}$.

Then $\lim_{n\to\infty} b_n = L$.

§15 Divergent Sequences

From now on, consider $a_n : \mathbb{N} \to \overline{\mathbb{R}}$.

<u>Def</u>:

- a_n <u>diverges to $+\infty$</u> if, $\forall M \in \mathbb{R}$, $\exists N$ s.t. $n \ge N$ ensures $a_n > M$.
  We write $\lim_{n\to\infty} a_n = +\infty$.

- a_n diverges to $-\infty$ if, $\forall M \in \mathbb{R}$, $\exists N$ s.t. $n \geq N$ ensures $a_n < M$. We write $\lim_{n\to\infty} a_n = -\infty$.

Def: Given a sequence a_n, the limit exists if a_n converges or a_n diverges to $\pm\infty$.

$\Updownarrow$

$\lim_{n\to\infty} a_n \in \overline{\mathbb{R}}$.

Thm (~~Squeeze~~): Suppose $a_n \leq b_n$ for all but finitely many n and the limits exist. Then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

16 Monotone Sequences and e

Thm: All bounded monotone sequences converge.

Ex: If $|a|<1$, $\lim_{n\to\infty} a^n = 0$.

Other useful facts... will define properly soon.

- If $a>0$, $\underbrace{a^{1/n} \longrightarrow 1}_{\text{"converges as } n\to+\infty \text{ to\_"}}$

(will return to $n^{1/n}$ ...)

Notation:

- If a_n is increasing and converges to L, $a_n \nearrow L$
- If a_n is decreasing and converges to L, $a_n \searrow L$

<u>Thm:</u>

- All increasing sequences that are unbounded above diverge to $+\infty$.
- All decreasing sequences that are unbounded below diverge to $-\infty$.

<u>Pf</u>: Suppose a_n is an increasing sequence that is unbounded above. Fix $M \in \mathbb{R}$. Then $\exists\ N \in \mathbb{N}$ s.t. $a_N > M$. Thus $n \geq N$, $a_n \geq a_N > M$. Hence $\lim_{n \to \infty} a_n = +\infty$.

If b_n is a decreasing sequence that is unbounded below,

$-b_n$ satisfies hyp of first bullet. Thus $\lim_{n\to\infty} -b_n = +\infty$.

Fix $M \in \mathbb{R}$. Then $\exists N \in \mathbb{N}$ s.t. $n \geq N$
$-b_n > -M \iff b_n < M$.
Hence $\lim_{n\to\infty} b_n = -\infty$. $\square$

Rmk:

- If a_n is increasing and not identically $-\infty$, then unbdd $\iff$ unbdd above
- If a_n is decreasing and not identically $+\infty$, then unbdd $\iff$ unbdd below

Rmk: Suppose a_n is monotone

1) a_n bdd $\Rightarrow$ converges

2) a_n unbdd

A. a_n increasing

(a) $a_n \equiv -\infty \Rightarrow \lim a_n = -\infty$

(b) unbdd above $\Rightarrow \lim a_n = +\infty$

B. a_n decreasing

(a) $a_n \equiv +\infty \Rightarrow \lim a_n = +\infty$

(b) unbdd below $\Rightarrow \lim a_n = -\infty$

Cor: For any monotone sequence $a_n : \mathbb{N} \to \overline{\mathbb{R}}$, $\lim\limits_{n\to\infty} a_n$ exists!

Now: define e as limit of a monotone sequence.

Lemma: For $0 \le a < b$,

$$\frac{b^{n+1} - a^{n+1}}{b-a} < (n+1)\, b^n$$

Pf:

$$\frac{b^{n+1} - a^{n+1}}{b-a} = (b^n + ab^{n-1} + a^2 b^{n-2} + \dots + a^{n-1}b + a^n)$$

$$< \underbrace{b^n + b^n + \dots + b^n}_{n+1}$$

$$= (n+1)\, b^n \qquad \square$$

Thm: The sequence $\left(1+\frac{1}{n}\right)^n$ is increasing and convergent. The limit is denoted e.

Pf: First, we show increasing.

By lemma, for $0 \leq a < b$,

$$b^n\left[b-(n+1)(b-a)\right] < a^{n+1}$$

Take $a = 1+\frac{1}{n+1}$, $b = 1+\frac{1}{n}$ to obtain...

$$\left(1+\frac{1}{n}\right)^n \underbrace{\left[\left(1+\frac{1}{n}\right)-\cancel{(n+1)}\overbrace{\left(\frac{1}{n}-\frac{1}{n+1}\right)}^{\frac{1}{n\cancel{(n+1)}}}\right]}_{=1} < \left(1+\frac{1}{n+1}\right)^{n+1}$$

Now, we show bdd above.

Take $a = 1$, $b = 1+\frac{1}{2n}$. Then

$$\left(1+\frac{1}{2n}\right)^n \underbrace{\left[\left(1+\frac{1}{2n}\right)-(n+1)\left(\frac{1}{2n}\right)\right]}_{1/2} < 1$$

This gives

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{2n}\right)^{2n} < 4.$$

increasing sequence

Thus the sequence is convergent. $\square$

Rmk: $2 \leq \left(1+\frac{1}{n}\right)^n < 4 \quad \forall n \in \mathbb{N}$

Thus $e \in [2,4]$.

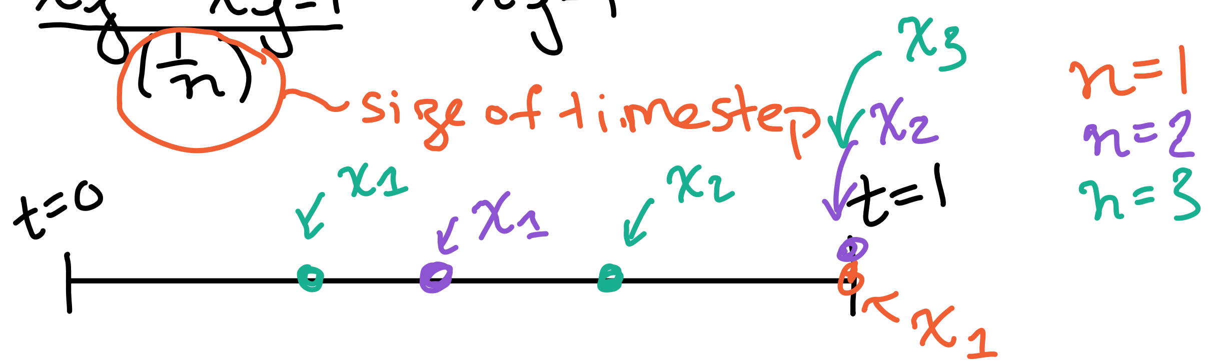
Warmup Dynamic Interpretation

$x_0 = 1$

For $j = 1, \dots, n$ # of time steps

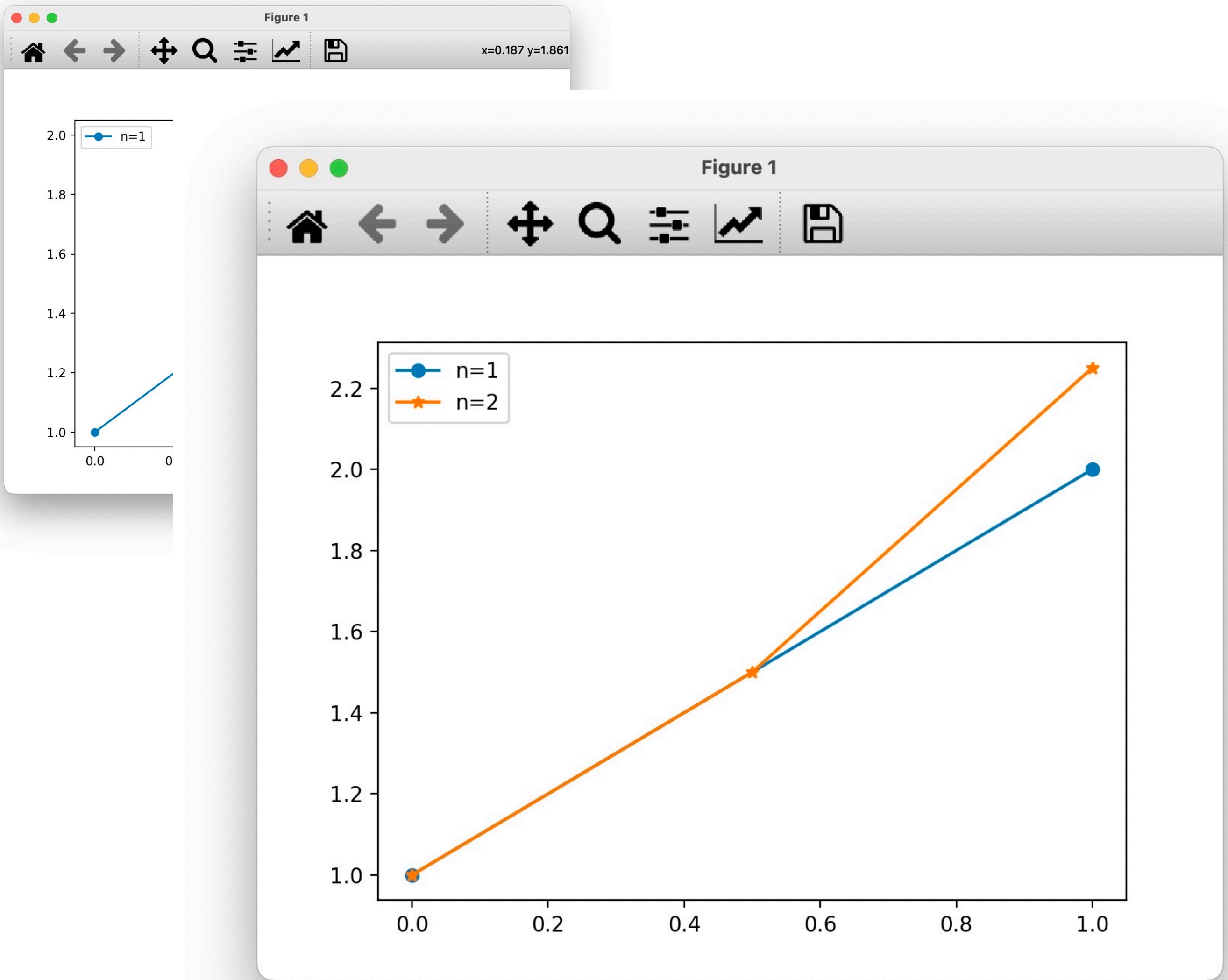
$$\frac{x_j - x_{j-1}}{\left(\frac{1}{n}\right)} = x_{j-1}$$

size of timestep



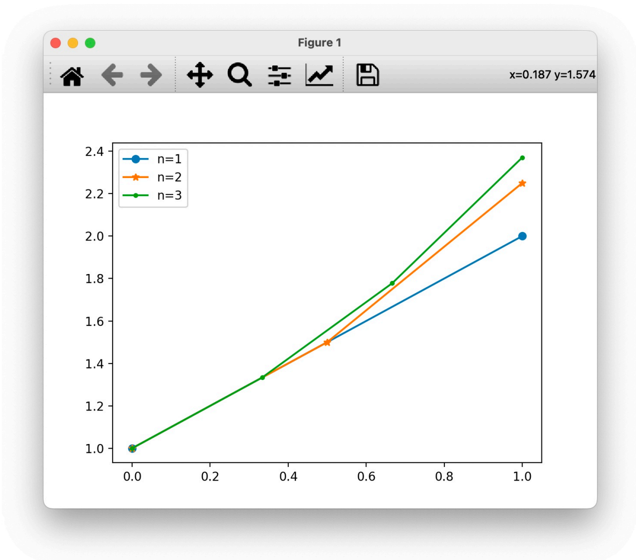
$$x_j = x_{j-1} + \frac{1}{n} x_{j-1} = \left(1 + \frac{1}{n}\right) x_{j-1}$$

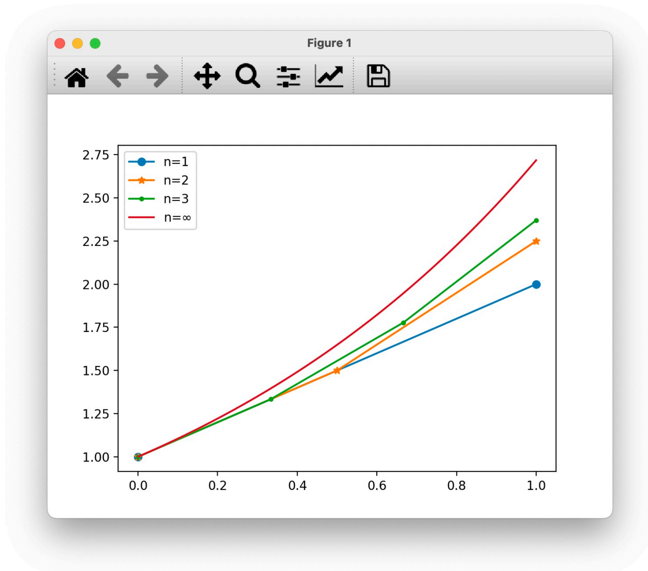
$$x_n = \left(1 + \frac{1}{n}\right)^n x_0 = \left(1 + \frac{1}{n}\right)^n$$



Continuum limit:

$$\left.\begin{cases} \dfrac{dx(t)}{dt} = x(t) \\ x(0) = 1 \end{cases}\right\} \to \begin{array}{l} \text{solution} \\ x(t) = e^{t} \end{array}$$





17 Real Exponents

Goal: For $a>0$, $x \in \mathbb{R}$, define a^x.

Background: rational exponents

Thm: $\forall a \geq 0$, $n \in \mathbb{N}$, there exists $b \geq 0$ s.t. $b^n = a$.
Denote b as $a^{1/n}$.

Pf: See J&P Thm 7.5
Similar to case $n=2$ on HW.

Cor: $\forall a \in \mathbb{R}$, $n \in \mathbb{N}$ odd, $\exists$ $b \in \mathbb{R}$ s.t. $b^n = a$. Denote b as $a^{1/n}$.

Pf: By previous thm, $\exists\, c \in \mathbb{R}$ s.t. $c^n = |a|$. If $a \geq 0$, $c^n = |a| = a$, and we are done.
If $a < 0$, let $b = -c$.
Then $b^n = (-c)^n = (-1)^n c^n = -c^n = a$.

n odd

$\square$

with no common factors

Def: For any $r \in \mathbb{Q}$, $\exists\, m \in \mathbb{Z}$, $n \in \mathbb{N}$ s.t. $r = \frac{m}{n}$. Define

$$x^r = (x^{1/n})^m,$$

for all x s.t. $x^{1/n}$ is defined.

Thm: For all $p, q \in \mathbb{Q}$, $x \in \mathbb{R}$,
If $x > 0$,
(i) $x^{p+q} = x^p x^q$
(ii) $x^p = \frac{1}{x^{-p}}$
(iii) $(xy)^p = x^p y^p$
(iv) $(x^p)^q = x^{pq}$

ensures definition of x^r is indep of expression of $r = \frac{m}{n}$, $m \in \mathbb{Z}, n \in \mathbb{N}$

(v) If $0 < x < y$, $p > 0$, then $x^p < y^p$.
(vi) If $x > 1$ and $p < q$, then $x^p < x^q$.

no common factors

Pf: Let $p = \frac{m}{n}$, $q = \frac{k}{\ell}$, $m, k \in \mathbb{Z}$ and $n, \ell \in \mathbb{N}$.

We will show (i).

$$x^{p+q} = x^{\frac{ml+kn}{ln}}$$

defn of rational exp is indep of rep of rational number

$$= \left(x^{\frac{1}{ln}}\right)^{ml+kn}$$

$$= \left(x^{\frac{1}{ln}}\right)^{ml} \left(x^{\frac{1}{ln}}\right)^{kn}$$

$$= x^{m/n} x^{k/l}$$

$$= x^p x^q$$