Lecture 8

Practice Midterm 1 Posted
(not to be turned in)

Notation:

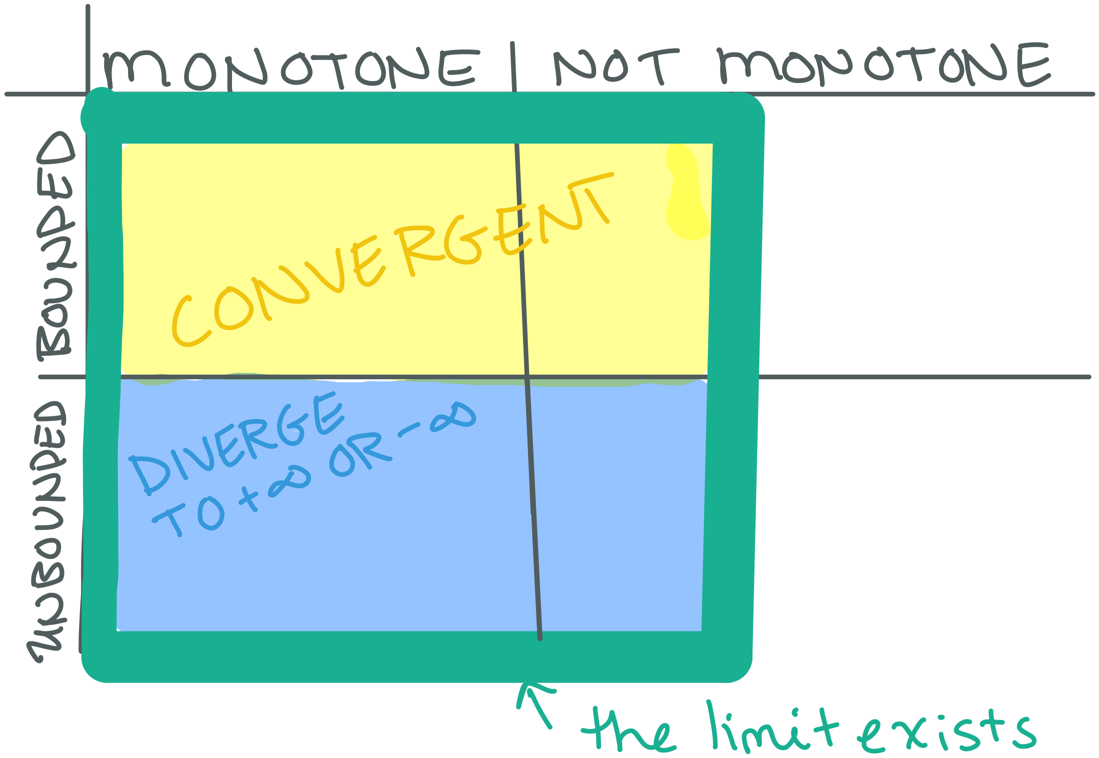
- If a_n is increasing and converges to L, $a_n \nearrow L$
- If a_n is decreasing and converges to L, $a_n \searrow L$

likewise $a_n \nearrow +\infty$, $a_n \searrow -\infty$.

Thm:

- All increasing sequences that are unbounded above diverge to $+\infty$.
- All decreasing sequences that are unbounded below diverge to $-\infty$.

For a real valued sequence s_n...



Rmk: Suppose $a_n : \mathbb{N} \to \overline{\mathbb{R}}$

- If a_n is increasing and not identically $-\infty$, then unbdd $\Longleftrightarrow$ unbdd above $\quad a_n = (-\infty, 1, 1, 1, \ldots)$
- If a_n is decreasing and not identically $+\infty$, then unbdd $\Longleftrightarrow$ unbdd below

Rmk: Suppose a_n is monotone

1) a_n bdd $\Rightarrow$ converges
2) a_n unbdd

   A. a_n increasing:

   (a) $a_n \equiv -\infty \Rightarrow \lim a_n = -\infty$

   (b) unbdd above $\Rightarrow \lim a_n = +\infty$

   B. a_n decreasing

   (a) $a_n \equiv +\infty \Rightarrow \lim a_n = +\infty$

   (b) unbdd below $\Rightarrow \lim a_n = -\infty$

Rmk: Suppose $a_n : \mathbb{N} \to \overline{\mathbb{R}}$ increasing

1) $a_n = -\infty \ \forall n \in \mathbb{N} \Rightarrow \boxed{\lim_{n\to\infty} a_n = -\infty}$

2) $a_n = -\infty$ for at most finitely many $n \in \mathbb{N}$

Then, up to modifying finitely many elts in sequence, a_n is bounded below

A. a_n is bounded above $\Rightarrow \boxed{\lim_{n\to\infty} a_n \in \mathbb{R}}$

B. a_n is unbounded above $\Rightarrow \boxed{\lim_{n\to\infty} a_n = +\infty}$

Similarly for $a_n : \mathbb{N} \to \overline{\mathbb{R}}$ decreasing

Cor: For $a_n : \mathbb{N} \to \overline{\mathbb{R}}$ monotone, $\lim_{n\to\infty} a_n$ exists.

Thm: The sequence $\left(1+\frac{1}{n}\right)^n$ is increasing and convergent. The limit is denoted e.

Recall:
- If $|a|<1$, $\lim_{n\to\infty} a^n = 0$
- If $a>0$, $\lim_{n\to\infty} a^{1/n} = \underline{1}$.
- $\forall n \in \mathbb{N}$, $a,b>0$
  $a<b \iff a^n < b^n$

Ex: For $n \geq 4$, the sequence $n^{1/n} \searrow 1$

To see this, note that

$$(n+1)^{\frac{1}{n+1}} \leq n^{\frac{1}{n}} \iff (n+1)^{n} \leq n^{n+1}$$
$$\iff \left(\frac{n+1}{n}\right)^{n} \leq n$$
$$\iff \left(1+\frac{1}{n}\right)^{n} \leq n$$

Thus this holds for all $n \geq 4$

Recall from previous thm that $\left(1+\frac{1}{n}\right)^{n} \leq 4 \quad \forall n \in \mathbb{N}$.

Furthermore $n^{1/n} \geq 0 \ \forall n \in \mathbb{N}$, so $n^{1/n}$ is bounded, hence converges to some $L \in \mathbb{R}$. Thus $(2k)^{1/2k}$ converges to L.

$$L^2 = \lim_{k\to\infty} (2k)^{2/2k} = \lim_{k\to\infty} \overbrace{2^{2/2k}}^{1} \left(\frac{2k}{2}\right)^{2/2k}$$
$$= \lim_{k\to\infty} 2^{1/k} (k)^{1/k} = L.$$

Furthermore, $1 \leq n \iff \underset{\substack{\| \\ 1}}{1^{1/n}} \leq n^{1/n}$

Thus $L=0$ is impossible. Hence $L=1$.

17 Real Exponents

<u>Goal</u>: For $a>0$, $x\in\mathbb{R}$, define a^x.
Background: rational exponents

<u>Thm</u>: $\forall a\geq 0, n\in\mathbb{N}$, there exists $b\geq 0$ s.t. $b^n=a$.
Denote b as $a^{1/n}$.

<u>Cor</u>: $\forall a\in\mathbb{R}$, $n\in\mathbb{N}$ odd, $\exists$ $b\in\mathbb{R}$ s.t. $b^n=a$. Denote b as $a^{1/n}$.

<u>Def</u>: For any $r\in\mathbb{Q}$, suppose $r=\frac{m}{n}$ for $m\in\mathbb{Z}, n\in\mathbb{N}$ is its expression in lowest terms. Define

$$x^r=(x^{1/n})^m,$$

for all x s.t. $x^{1/n}$ is defined.

<u>Lemma</u>: For all $x>0$, $r\in\mathbb{Q}$, if $r=\underbrace{\frac{k}{\ell}}_{\text{not necessarily in lowest terms}}$, $k\in\mathbb{Z}$, $\ell\in\mathbb{N}$, then

$$x^r=\left(x^{1/\ell}\right)^k$$

Pf: First, note that for any $i,j\in\mathbb{N}$ $(x^i)^j=x^{ij}$. Thus,

$$\left(\left(x^{\frac{1}{j}}\right)^{\frac{1}{i}}\right)^{ij}=\left(\left(\left(x^{\frac{1}{j}}\right)^{\frac{1}{i}}\right)^{i}\right)^{j}=x$$

Thus, $\left(x^{\frac{1}{j}}\right)^{\frac{1}{i}}=x^{\frac{1}{ij}}$.

Let $r=\frac{m}{n}$ be the expression of r in lowest terms. Then $\frac{m}{n}=\frac{k}{\ell}\iff k=m\frac{\ell}{n}$.

Thus, n is a divisor of l,
so $\exists\, j \in \mathbb{N}$ s.t. $l = nj$.
Likewise $k = mj$.

Then

$$(x^{1/l})^k = (x^{\frac{1}{nj}})^{mj} = \left(\left((x^{\frac{1}{n}})^{\frac{1}{j}}\right)^j\right)^m$$

$$= x^{\frac{m}{n}} = x^r.$$

<u>Thm</u>: For all $p, q \in \mathbb{Q}$, $x \in \mathbb{R}$,
If $x > 0$,
(i) $x^{p+q} = x^p x^q$
(ii) $x^p = \frac{1}{x^{-p}}$
(iii) $(xy)^p = x^p y^p$
(iv) $(x^p)^q = x^{pq}$

(v) If $0 < x < y$, $p > 0$, then $x^p < y^p$.
(vi) If $x > 1$ and $p < q$, then $x^p < x^q$.

Pf: Suppose $p = \frac{m}{n}$, $q = \frac{k}{l}$ for $m, k \in \mathbb{Z}$, $n, l \in \mathbb{N}$.

First, we will show (i).

$$x^{p+q} = x^{\frac{ml+kn}{ln}}$$
$$= \left(x^{\frac{1}{ln}}\right)^{ml+kn}$$
$$= \left(x^{\frac{1}{ln}}\right)^{ml} \left(x^{\frac{1}{ln}}\right)^{kn}$$
$$= x^{m/n}\, x^{k/l}$$
$$= x^p x^q$$

Next, we show (v). Since $\overbrace{x<y}^{a^n<b^n}$
$\Longrightarrow \underbrace{x^{\frac{1}{n}} < y^{\frac{1}{n}}}_{a<b} \Longrightarrow x^{\frac{m}{n}} < y^{\frac{m}{n}}$. $\square$

Now: real valued exponents.

Idea: For any $a>0$, define a^x as $\lim_{n\to\infty} a^{r_n}$, where $r_n : \mathbb{N} \to \mathbb{Q}$ satisfying $\lim_{n\to\infty} = x$.

Thm: $\forall x \in \mathbb{R}$, $\exists\, r_n : \mathbb{N} \to \mathbb{Q}$ s.t. $r_n \nearrow x$.

Pf: Choose $r_1 \in \mathbb{Q}$ s.t. $x-1 < r_1 < x$. Suppose we have defined $(r_n)_{n=1}^{k-1}$ to be an increasing sequence s.t. $x - \frac{1}{n} < r_n < x$, $\forall n = 1, \ldots, k-1$.

Choose $r_k \in \mathbb{Q}$ so that
$$\max\{x - \tfrac{1}{k}, r_{k-1}\} < r_k < x$$

By construction r_n is increasing

and since $x-\frac{1}{n}<r_n<x$ $\forall n\in\mathbb{N}$, by Squeeze Thm, $\lim_{n\to\infty} r_n = x$. $\square$

Lemma: Suppose $a>1$, $x\in\mathbb{R}$ and $r_n, s_n:\mathbb{N}\to\mathbb{Q}$ s.t. $r_n\nearrow x$, $s_n\nearrow x$. Then $\lim_{n\to\infty} a^{r_n} = \lim_{n\to\infty} a^{s_n}$.

Pf: First, observe that by previous thm, part (vi), a^{r_n} is increasing. Also $1<a^{r_n}<a^{y_0}$ for $y_0\in\mathbb{Q}$, $y_0>x$. Thus a^{r_n} converges. Similarly a^{s_n} converges.

Define $R_n = r_n - \frac{1}{n}$ and $S_n = s_n - \frac{1}{n}$. Then

$$\lim_{n\to\infty} a^{R_n} = \lim_{n\to\infty} a^{r_n - \frac{1}{n}} = \lim_{n\to\infty} a^{r_n} \overbrace{a^{-\frac{1}{n}}}^{\to 1}$$
$$= \lim_{n\to\infty} a^{r_n}$$

and similarly $\lim_{n\to\infty} a^{S_n} = \lim_{n\to\infty} a^{s_n}$.

Thus, it suffices to show

$$\lim_{n\to\infty} a^{R_n} = \lim_{n\to\infty} a^{S_n}.$$

We construct a new sequence $b_k \nearrow x$ as follows. Let $b_1 = R_1$.
Since $R_1 < x$, $\exists\, n_2$ s.t.
$R_1 < \underbrace{S_{n_2}}_{= b_2} < x$. Likewise, $\exists\, n_3$

s.t. $n_3 > 1$ and $\underset{\overset{\|}{b_3}}{R_{n_3}} > S_{n_2}$.

Finally, $\exists\, n_4 > n_2$ s.t. $S_{n_4} > R_{n_3}$.
In this way, we construct
b_k so odd elts are subseq
of R_n and even elts are
subseq of S_n. Thus $b_k \nearrow x$.

So $\lim_{k\to\infty} a^{b_k} \in \mathbb{R}$.

Since all subseq must have
same limits. $\square$