MATH CCS 117: MIDTERM 1
Monday, May 6, 2024

Name: ____________________________________________

Signature: _________________________________________

This is a closed-book and closed-note examination. Please show your work in the space provided. You may use scratch paper. You have 1 hour and 15 minutes.

<table>
<thead>
<tr>
<th>Question</th>
<th>Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>extra credit</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>30</td>
<td></td>
</tr>
</tbody>
</table>
Question 1 (10 points)

Consider a sequence \( a_n \) satisfying \( a_n \neq 0 \), for all but finitely many \( n \in \mathbb{N} \). If \( \lim_{n \to +\infty} \frac{a_{n+1}}{a_n} = 0 \), find \( \lim_{n \to +\infty} a_n \).

We will show that \( \lim_{n \to +\infty} a_n = 0 \).

For any \( \epsilon' > 0 \), there exists \( N' \) s.t. \( n \geq N' \) ensures \( |\frac{a_{n+1}}{a_n}| < \epsilon' \). Thus \( |a_{n+1}| < \epsilon'|a_n| \), and so \( |a_{n+m}| < \epsilon'|a_{n+m-1}| < \epsilon'|a_{n+m-2}| < \cdots < \epsilon'|a_N| \), for all \( m \geq N \).

If \( \epsilon' < 1 \), \( \lim_{m \to \infty} \epsilon'^m = 0 \) \( \implies \lim_{m \to \infty} \epsilon'^m |a_N| = 0 \). Let \( \epsilon' = \frac{1}{2} \), and choose \( N' \) as above.

Fix \( \epsilon > 0 \) arbitrary. Since \( \lim_{m \to \infty} (\frac{1}{2})^m |a_N| = 0 \), there exists \( M \) s.t. \( m \geq M \) ensures \( |(\frac{1}{2})^m |a_N|| < \epsilon \). Thus, if \( n \geq M + N' \), then \( n = N' + m \) for some \( m \geq M \) and \( |a_n| = |a_{N' + m}| < (\frac{1}{2})^m |a_N| < \epsilon \).

This shows \( \lim_{n \to +\infty} a_n = 0 \).
Question 2 (10 points)

In class, we proved that, if $-1 < a < 1$, then $\lim_{n \to +\infty} a^n = 0$. Furthermore, it is clear that, if $a = 1$, then $\lim_{n \to +\infty} a^n = 1$.

(a) If $a > 1$, prove that $\lim_{n \to +\infty} a^n = +\infty$.

(b) If $a < -1$, prove that the limit of $a^n$ does not exist.

Note that $a > 1$ ensures $a^n < a^{n+1}$, so the sequence is strictly increasing. Thus, it suffices to prove $a^n$ is unbounded above. Assume, for the sake of contradiction that $a^n$ is bounded above. Then it must converge to some $L \in \mathbb{R}$. Thus $\lim_{n \to \infty} a^n = \lim_{n \to \infty} a^{n+1} = \lim_{n \to \infty} a^n a = L a$. Since a sequence of convergent subsequences has some limit $L = 0$ is impossible since $a^n$ is strictly increasing and $a > 1$. Thus $L = L a$ implies $L = 1$, which is a contradiction. This shows $a^n$ is unbounded above.

(b) First, we show $\lim_{n \to \infty} a^n \neq \pm \infty$.

Since the odd elements are negative and the even elements are positive, for $m = 0$, there does not exist $N$ s.t. $n \geq N$ ensures either $a^n \geq M$ or $a^n \leq M$. Thus $\lim_{n \to \infty} a^n \neq \infty$.
Now, we show $a^n$ does not converge.

By the previous part, $\lim_{n \to \infty} 1_{a^n} = +\infty$.

Thus, $a^n$ is not a bounded sequence.

Hence, it cannot converge.
First, we show (a). Suppose $S$ is unbounded above, so $\sup(S) = +\infty$. We must show $kS$ is unbounded above. Since $S$ is unbounded above, $\forall M \in \mathbb{R}$, $\exists s \in S$ s.t. $s \geq \frac{M}{k} \Rightarrow ks \geq M$. Thus $kS$ is unbounded above.

Now suppose $S$ is unbounded above. Since $\sup(s)$ is an upper bound for $S$, $s \leq \sup(s) \forall s \in S \Rightarrow ks \leq k\sup(s) \forall s \in S \Rightarrow k\sup(s)$ is an upper bound for $kS$. Suppose $M$ is an upper bound for $kS$, that is, $ks \leq M \forall s \in S$. Then $\frac{M}{k}$ is an upper bound for $S$, so $\frac{M}{k} \geq \sup(S) \Rightarrow M \geq k\sup(S)$. This shows $k\sup(S)$ is the least upper bound of $kS$. Hence $\sup(kS) = k\sup(S)$. 

(a) If $k > 0$, prove that $\sup(kS) = k\sup(S)$.

(b) If $k < 0$, prove that $\sup(kS) = k\inf(S)$. 

Given a nonempty subset $S \subseteq \mathbb{R}$ and $k \in \mathbb{R}$, define $kS := \{ks : s \in S\}$. (Note that $S$ can be any nonempty subset; it is not necessarily bounded above.)