MATH CCS 117: MIDTERM 1

Monday, May 6, 2024

Name: ______________________________

Signature: ______________________________

This is a closed-book and closed-note examination. Please show your work in the space provided. You may use scratch paper. You have 1 hour and 15 minutes.

Question	Points	Score
1	10	
2	10	
3	10	
4	extra credit	
Total	30	

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Question 1 (10 points)

Consider a sequence~~s~~ a_n satisfying $a_n \neq 0$, for all but finitely many $n \in \mathbb{N}$. If $\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = 0$, find $\lim_{n\to+\infty} a_n$.

We will show that $\lim_{n\to\infty} a_n = 0$.

For any $\varepsilon' > 0$, there exists N' s.t. $n \geq N'$ ensures, $a_n \neq 0$ and $\left|\frac{a_{n+1}}{a_n}\right| < \varepsilon'$. Thus $|a_{n+1}| < \varepsilon' |a_n|$,

so $|a_{N'+m}| < \varepsilon' |a_{N'+m-1}| < (\varepsilon')^2 |a_{N'+m-2}| < \dots < (\varepsilon')^m |a_{N'}|$, for all $m \in \mathbb{N}$.

If $\varepsilon' < 1$, $\lim_{m\to\infty} (\varepsilon')^m = 0 \Rightarrow \lim_{m\to\infty} (\varepsilon')^m |a_{N'}| = 0$.

Let $\varepsilon' = \frac{1}{2}$, and choose N' as above.

Fix $\varepsilon > 0$ arbitrary. Since $\lim_{m\to\infty} (\frac{1}{2})^m |a_{N'}| = 0$, there exists M s.t. $m \geq M$ ensures $|(\frac{1}{2})^m |a_{N'}|| < \varepsilon$. Thus, if $n \geq M + N'$, then $n = N' + m$ for some $m \geq M$ and

$$|a_n| = |a_{N'+m}| < (\tfrac{1}{2})^m |a_{N'}| < \varepsilon.$$

This shows $\lim_{n\to\infty} a_n = 0$.

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Question 2 (10 points)

In class, we proved that, if $-1 < a < 1$, then $\lim_{n\to+\infty} a^n = 0$. Furthermore, it is clear that, if $a = 1$, then $\lim_{n\to+\infty} a^n = 1$.

(a) If $a > 1$, prove that $\lim_{n\to+\infty} a^n = +\infty$.

(b) If $a < -1$, prove that the limit of a^n does not exist.

(a) Note that $a>1$ ensures $a^n < a^{n+1}$, so the sequence is strictly increasing.

Thus, it suffices to prove a^n is unbounded above. Assume, for the sake of contradiction that a^n is bounded above. Then it must converge to some $L \in \mathbb{R}$. Thus

$$L = \lim_{n\to\infty} a^n = \lim_{n\to\infty} a^{n+1} = \lim_{n\to\infty} a^n a = La.$$

subsequence of convergent sequence has same limit

$L=0$ is impossible since a^n is strictly increasing and $a>1$. Thus $L=La$ implies $a=1$, which is a contradiction. This shows a^n is unbounded above.

(b) First, we show $\lim_{n\to\infty} a^n \neq \pm\infty$.

Since the odd elements are negative and the even elements are positive, for $M=0$, there does not exist N s.t. $n \geq N$ ensures either $a^n \geq M$ or $a^n \leq M$. Thus $\lim_{n\to\infty} a^n \neq \pm\infty$.

Now, we show a^n does not converge.

By the previous part, $\lim_{n\to\infty} \underbrace{|a^n|}_{=|a|^n} = +\infty$.

Thus a^n is not a bounded sequence.

Hence, it cannot converge.

Question 3 (10 points)

Given a nonempty subset $S \subseteq \mathbb{R}$ and $k \in \mathbb{R}$, define $kS := \{ks : s \in S\}$. (Note that S can be *any* nonempty subset; it is not necessarily bounded above.)

(a) If $k > 0$, prove that $\sup(kS) = k\sup(S)$.

~~(b) If $k < 0$, prove that $\sup(kS) = k\inf(S)$.~~

First, we show (a). Suppose S is unbounded above, so $k\sup(S) = +\infty$. We must show kS is unbounded above. Since S is unbounded above, $\forall\, m \in \mathbb{R}$, $\exists\, s \in S$ s.t. $s \geq \frac{m}{k}$ $\Rightarrow ks \geq m$. Thus kS is unbounded above.

Now suppose S is unbounded above. Since $\sup(S)$ is an upper bound for S, $s \leq \sup(S)$ $\forall\, s \in S \Rightarrow ks \leq k\sup(S)$ $\forall\, s \in S \Rightarrow k\sup(S)$ is an upper bound for kS. Suppose m is an upper bound for kS, that is, $ks \leq m$ $\forall\, s \in S$. Then $\frac{m}{k}$ is an upper bound for S, so $\frac{m}{k} \geq \sup(S) \Rightarrow m \geq k\sup(S)$. This shows $k\sup(S)$ is the least upper bound of kS. Hence $\sup(kS) = k\sup(S)$.