Math CCS 117: Midterm 2

Wednesday May 29, 2024

Name: ______________________________

Signature: ______________________________

This is a closed-book and closed-note examination. Please show your work in the space provided. You may use scratch paper. You have 1 hour and 15 minutes.

Question	Points	Score
1	12	
2	8	
3	10	
4	extra credit	
Total	30	

Question 1 (12 points)

(a) Prove the following by induction: for $a \neq 1$,

$$\sum_{i=0}^{m-1} a^i = \frac{1-a^m}{1-a}.$$

(b) Use part (a) to show that

$$\sum_{i=n}^{m-1} a^i = \frac{a^n - a^m}{1-a}.$$

(Hint: $\sum_{i=n}^{m-1} a^i = \sum_{i=0}^{m-1} a^i - \sum_{i=0}^{n-1} a^i$.)

(c) Recall that, by the triangle inequality, for any $x_n : \mathbb{N} \to \mathbb{R}$ we may estimate

$$\left|\sum_{j=1}^{N} x_j\right| = |x_1 + x_2 + \cdots + x_N| \leq |x_1| + |x_2| + \cdots + |x_N| = \sum_{j=1}^{N} |x_j|.$$

Let s_n, $s_n : \mathbb{N} \to \mathbb{R}$ be a sequence such that $|s_{n+1} - s_n| \leq 4^{-n}$ for all $n \in \mathbb{N}$. Use part (b) and the above inequality to prove s_n is a Cauchy sequence.

(d) Does the sequence from part (c) converge? Justify your answer.

(a). We proceed by induction. Base case: For $m=1$, $\sum_{i=0}^{0} a^i = 1 = \frac{1-a}{1-a}$ for $a \neq 1$. Inductive step: Suppose $\sum_{i=0}^{m-1} a^i = \frac{1-a^m}{1-a}$. Then $\sum_{i=0}^{m} a^i = \frac{1-a^m}{1-a} + a^m$ $= \frac{1-a^m + a^m(1-a)}{1-a} = \frac{1-a^{m+1}}{1-a}$, which gives the result.

(b) Following the hint, $\sum_{i=n}^{m-1} a^i = \sum_{i=0}^{m-1} a^i - \sum_{i=0}^{n-1} a^i \overset{(a)}{=} \frac{1-a^m}{1-a} - \frac{1-a^n}{1-a} = \frac{a^n - a^m}{1-a}$

(c) Fix $\varepsilon > 0$. We must show that $\exists N \in \mathbb{N}$ s.t. $m,n \geq N$ ensures $|s_n - s_m| < \varepsilon$. This is always true for $m=n$, so WLOG we may suppose $m > n$.

By part (b) and the triangle inequality,

$$|s_m - s_n| = \left|\sum_{i=n}^{m-1} s_{i+1} - s_i\right| \leq \sum_{i=n}^{m-1} |s_{i+1} - s_i| \leq \sum_{i=n}^{m-1} 4^{-i} = \frac{4^{-n} - 4^{-m}}{\frac{3}{4}}$$

Fix $\varepsilon > 0$. Since $\left(\frac{1}{4}\right) \in (0,1)$, $\exists$ N s.t. $n \geq N$ ensures $\frac{4}{3}(4^{-n} - 4^{-m}) \leq \frac{4}{3} 4^{-n} = \frac{4}{3}\left(\frac{1}{4}\right)^n < \varepsilon$.

Thus, $m > n \geq N$ ensures $|s_m - s_n| < \varepsilon$, so s_n is a Cauchy sequence.

(d) Yes. All Cauchy sequences are convergent.

Question 2 (8 points)

(a) Suppose s_n has a subsequence s_{n_k} that is bounded. Explain why this implies that s_n has a convergent subsequence.

(b) Suppose that s_n has no convergent subsequences. Prove that $\lim_{n\to+\infty} |s_n| = +\infty$.
(Hint: prove the result by contradiction, using part (a).)

(a) By Bolzano-Weierstrass, s_{n_k} has a subsequence $s_{n_{k_\ell}}$ that converges. Since $s_{n_{k_\ell}}$ is a subsequence of s_n, this gives the result.

(b) Assume, for the sake of contradiction, that $\lim_{n\to\infty} |s_n| \neq +\infty$. Then $\exists\, M \in \mathbb{R}$ s.t. $\forall N \in \mathbb{N}$, $\exists\, n \geq N$ s.t. $|s_n| < M$.

Thus, we may choose $n_1 \in \mathbb{N}$ s.t. $|s_{n_1}| < M$. Assume we have chosen $n_k \in \mathbb{N}$, $n_k > n_{k-1}$, so that $|s_{n_k}| < M$. Then $\exists\, n_{k+1} \geq n_k + 1 > n_k$ s.t. $|s_{n_{k+1}}| < M$. Thus, s_{n_k} is a bounded subsequence of s_n. Hence, by part (a), s_n must have a convergent subsequence. This is a contradiction. Thus,

$\lim_{n\to\infty} |s_n| = +\infty$.

Question 3 (10 points)

(a) ~~(i)~~ Consider the signum function $\text{sgn} : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by

$$\text{sgn}(x) = \frac{|x|}{x}.$$

For all $x \in \mathbb{R} \setminus \{0\}$, prove that sgn is continuous at x.

(b) ~~(ii)~~ Fix $a \in \mathbb{R}$ arbitrary, and consider the function $g_a : \mathbb{R} \to \mathbb{R}$ defined by

$$g_a(x) = \begin{cases} \frac{|x|}{x} & \text{for } x \neq 0, \\ a & \text{for } x = 0. \end{cases}$$

Prove that g_a is not continuous at all $x \in \mathbb{R}$.

(a) Fix $x_0 \in \mathbb{R} \setminus \{0\}$. Fix $\varepsilon > 0$. Since $x_0 \in \mathbb{R} \setminus \{0\}$, $\delta = |x_0| > 0$. Then $\forall\, x \in \mathbb{R} \setminus \{0\}$ with $|x - x_0| < \delta$, we either have $x, x_0 \in (0, +\infty)$ or $x, x_0 \in (-\infty, 0)$. In both cases $|\text{sgn}(x) - \text{sgn}(x_0)| = 0 < \varepsilon$. Thus sgn is continuous at x_0.

(b) We will show g_a is not continuous at $x_0 = 0$. Let $\varepsilon = \min\{|1-a|, |-1-a|\}$. Then either $|1-a| \not< \varepsilon$ or $|-1-a| \not< \varepsilon$ or both. Case 1: $\varepsilon > 0$. Then, for all $\delta > 0$, $x_1 = \delta/2$ and $x_2 = -\delta/2$ satisfy $|x - 0| < \delta$, but either $|g_a(x_1) - a| = |1-a| \not< \varepsilon$ or $|g_a(x_2) - a| = |-1-a| \not< \varepsilon$ or both. Thus g_a is not continuous at $x_0 = 0$. Case 2: $\varepsilon = 0$. Let $\tilde{\varepsilon} = \frac{1}{2}$. If $a = 1$, then for all $\delta > 0$, $x_1 = -\delta/2$ satisfies $|x_1 - 0| < \delta$ but $|g_a(x_1) - a| = 2 > \tilde{\varepsilon}$. If $a = -1$, then for all $\delta > 0$, $x_1 = \delta/2$ satisfies $|x_1 - 0| < \delta$ but $|g_a(x_1) - a| = 2 > \tilde{\varepsilon}$. Thus g_a is not cts at $x_0 = 0$.

Question 4 - Extra Credit

(i) Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$\lim_{h \to 0} f(x+h) - f(x-h) = 0.$$

Does this imply that f is continuous at x?

(ii) Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \in \mathbb{R}.$$

Does this imply that f is continuous at x?