Math CCS 117: Practice Midterm 1
(Not to be turned in)

Question 1
Define a sequence $s_n$ as follows: $s_1 = 1$ and, for $n \geq 1$, $s_{n+1} = \frac{1}{3}(s_n + 1)$. Find $\lim_{n \to \infty} s_n$.

Question 2
(a) Suppose $\lim_{n \to \infty} s_n = +\infty$ and $s_{n_k}$ is a subsequence of $s_n$. Prove that $\lim_{k \to \infty} s_{n_k} = +\infty$.
(b) Suppose $s_n$ is a sequence for which the limit does not exist—that is $s_n$ doesn’t converge or diverge to $\pm\infty$—and $s_{n_k}$ is a subsequence of $s_n$. Does the limit of $s_{n_k}$ not exist? Justify your answer with a proof or counterexample.

Question 3
Suppose $A$ and $B$ are nonempty subsets of $\mathbb{R}$. (Note that we do not assume that either $A$ or $B$ is bounded above.) Define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Prove $\sup(A + B) = \sup A + \sup B$.

Question 4 - Extra Credit
Given a sequence $s_n$ of real numbers, define its arithmetic mean by
$$\sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n}.$$ 
(a) If $s_n$ converges, prove that $\sigma_n$ converges.
(b) Give an example to show that the converse of part (a) is not true.
(c) Let $a_n = s_{n+1} - s_n$. Assume that $\lim_{k \to \infty} ka_k = 0$ and $\sigma_n$ converges. Prove that $s_n$ converges.

Hint: First, show that
$$s_{n+1} - \sigma_{n+1} = \frac{1}{n+1} \sum_{k=1}^{n} ka_k.$$ 
Moral of the problem: while the convergence of $\sigma_n$ is not, in general, sufficient to imply the convergence of $s_n$, if we also know that the increments of $s_n$ converge to zero sufficiently quickly, it is sufficient.