Math CCS 117: Practice Midterm 2

(Not to be turned in)

Question 1

Let $0 \leq \alpha < 1$, and let f be a function from $\mathbb{R}$ to $\mathbb{R}$ that satisfies

$$|f(x) - f(y)| \leq \alpha |x - y|, \quad \text{for all } x, y \in \mathbb{R}.$$

(Such a function is called an α-Lipschitz function.)

Let $a_1 \in \mathbb{R}$, and let $a_{n+1} = f(a_n)$ for $n \in \mathbb{N}$. Prove that a_n is a Cauchy sequence.

Question 2

This question will lead you through an alternative proof that any continuous function $f : [a, b] \to \mathbb{R}$ is bounded on $[a, b]$.

Suppose $f : [a, b] \to \mathbb{R}$ is a function that is *not* bounded on $[a, b]$.

(i) Prove that there exists $x_n : \mathbb{N} \to [a, b]$ so that $\lim_{n \to +\infty} |f(x_n)| = +\infty$.

(ii) Explain why we may choose the sequence x_n from part (i) so that it converges to some $x_0 \in [a, b]$.

(iii) Use part (ii) to show that the function f is not continuous at x_0.

Question 3

Fix $a \in \mathbb{R}$ and consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f_a(x) = \begin{cases} \frac{1+x}{1-x^2} & \text{if } x \notin \{-1, 1\} \\ a & \text{if } x \in \{-1, 1\}. \end{cases}$$

(i) Prove that there exists $a \in \mathbb{R}$ so that f_a is continuous at x, for all $x \in \mathbb{R} \setminus \{1\}$.

(ii) Show that, for all $a \in \mathbb{R}$, f_a is not continuous at 1.

Question 4 - Extra Credit

In this problem, you will prove part of the *Intermediate Value Theorem.*

THEOREM 1. *Consider an interval $I \subseteq \mathbb{R}$ and suppose $f : I \to \mathbb{R}$ is continuous at x, for all $x \in I$. Then, for any $a, b \in I$ with $a < b$ and $f(a) < f(b)$, if $y_0 \in (f(a), f(b))$, there exists $x_0 \in (a, b)$ so that $f(x_0) = y_0$.*

(i) Define $S := \{x \in [a, b] : f(x) < y_0\}$. Prove that $S \neq \emptyset$ and S is bounded above.

(ii) Define $x_0 := \sup(S)$. Show that $x_0 \in (a, b)$.

(iii) Show that there exists a sequence $s_n : \mathbb{N} \to S$ so that $\lim_{n\to+\infty} s_n = x_0$ and $y \geq \lim_{n\to+\infty} f(s_n) = f(x_0)$.

(iv) Show that there exists a sequence $t_n : \mathbb{N} \to [a,b] \setminus S$ so that $\lim_{n\to+\infty} t_n = x_0$ and $y \leq \lim_{n\to+\infty} f(t_n) = f(x_0)$.

(v) Explain why parts (b) and (c) ensure that $f(x_0) = y$.