Math CS 117: Homework 1

Due Monday, April 7th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1

Suppose F is a field and $a, b, c \in F$. Prove the following:

(a) $(-a)b = -ab$;

(b) $(-a)(-b) = ab$;

(c) $ac = bc$ and $c \neq 0$ implies $a = b$;

(d) $ab = 0$ implies either $a = 0$ or $b = 0$.

Question 2*

Suppose F is an ordered field and $a, b, \epsilon \in F$. Recall that, if 1 is the multiplicative identity of F, we define $2 := 1 + 1$. Prove the following:

(a) if $0 < a < b$, then $0 < 1/b < 1/a$;

(b) $2ab \leq a^2 + b^2$;

(c) if $a \leq b + \epsilon$ for all $\epsilon > 0$, then $a \leq b$.

We will use this last property repeatedly throughout the course.

Question 3

In the definition of a field, suppose that the condition $1 \neq 0$ in item M4 was removed. Prove that, if F is a field for which $1 = 0$, then $F = \{0\}$.

Note: From now on, you do not need to cite which individual properties of an ordered field (e.g. (A1), (A2), etc.) that you are using, and you may combine a few properties in a single logical step.

Question 4*

Consider the *Gaussian rational field* $\mathbb{Q}(i)$, defined to be the set

$$\mathbb{Q}(i) = \{p + qi : p, q \in \mathbb{Q}\},$$

where i denotes an element satisfying $i^2 = -1$. As the name implies, the Gaussian rational field is a field, endowed with the following addition and multiplication operations:

$$(p + qi) + (p' + q'i) = (p + p') + (q + q')i, \quad (p + qi) \cdot (p' + q'i) = (pp' - qq') + (pq' + qp')i.$$

The additive identity is $0 = 0 + 0i$ and the multiplicative identity is $1 = 1 + 0i$.

We may endow the Gaussian rational field with the *lexicographical ordering* given by

$$p + qi \leq p' + q'i \iff \text{either (i) } p < p' \text{ or (ii) } p = p' \text{ and } q \leq q'.$$

(This is sometimes known as the *dictionary ordering*, since it follows the same principle by which one puts a list of words in alphabetical order.)

(a) For any $x, y \in \mathbb{Q}(i)$ prove that exactly one of the following is true: $x < y$, $x = y$ or $x > y$.

(b) For any $x, y, z \in \mathbb{Q}(i)$, if $x \leq y$ and $y \leq z$, prove that $x \leq z$.

(c) Even though $\mathbb{Q}(i)$ is a field and can be endowed with an ordering as described above, it is *not* an ordered field. Which part of the definition of an ordered field is violated? Justify your answer.

Question 5

Suppose F is an ordered field and $a, b \in F$. Prove the following basic properties of the absolute value stated in class.

(a) $|a| \geq 0$

(b) $|ab| = |a|\ |b|$

(c) $|a| \geq a$ and $|a| \geq -a$

(d) $|a + b| \leq |a| + |b|$

Question 6*

Suppose F is an ordered field and $a, b \in F$. Prove the following properties of the absolute value.

(a) $|b| \leq a$ if and only if $-a \leq b \leq a$;

(b) $||a| - |b|| \leq |a - b|$.

The second inequality is known as the reverse triangle inequality.

Question 7*

Suppose F is an ordered field and $a, b, c \in F$. Prove the following properties of the absolute value. **We will use these inequalities repeatedly throughout the course.**

(a) Prove that $|a - b| \leq c$ if and only if $b - c \leq a \leq b + c$

(b) Prove that $|a - b| < c$ if and only if $b - c < a < b + c$.

Question 8

Let F be an ordered field, and let $0'$ and $1'$ denote the additive and multiplicative identities of F. For any $n \in \mathbb{N}$, let n' denote the element of F obtained by adding $1'$ to itself n times; that is,

$$n' := \sum_{i=1}^{n} 1' = \underbrace{1' + \cdots + 1'}_{n \text{ times}}.$$

Let $\mathbb{N}'$ denote the collection of all such elements in F,

$$\mathbb{N}' := \left\{ n' : n \in \mathbb{N} \right\} \subseteq F.$$

Define $f : \mathbb{N} \to \mathbb{N}'$ by $f(n) = n'$. Prove the following, for all $n, m, \in \mathbb{N}$:

(i) f is bijective;

(ii) $f(n+m) = f(n) + f(m)$;

(iii) $f(n) < f(m) \iff n < m$;

(iv) $f(nm) = f(n)f(m)$.

Question 9*

This question is a continuation of the previous question. For any $S \subseteq F$, let $-S = \{-s : s \in S\}$. Define

$$\mathbb{Z}' := (-\mathbb{N}') \cup \{0'\} \cup \mathbb{N}' \subseteq F.$$

Define a function $g : \mathbb{Z} \to \mathbb{Z}'$ so that $g(n) = f(n)$ for all $n \in \mathbb{N}$, where f is as in Question 8, and g satisfies properties (i)-(iv) from Question 8 for all $n, m \in \mathbb{Z}$. Justify your answer with a proof.

Question 10

This question is a continuation of the previous questions. Define

$$\mathbb{Q}' = \left\{ \frac{p'}{q'} : q' \in \mathbb{Z}' \setminus \{0'\}, p' \in \mathbb{Z}' \right\} \subseteq F.$$

Define a function $h : \mathbb{Q} \to \mathbb{Q}'$ by

$$h\left(\frac{p}{q}\right) = \frac{g(p)}{g(q)}, \text{ for all } q \in \mathbb{Z} \setminus \{0\},\ p \in \mathbb{Z},$$

where g is the function from the previous question.

(i) Prove that h is a well-defined function by showing that if $\frac{p}{q} = \frac{a}{b} \in \mathbb{Q}$ for $q, b \in \mathbb{Z} \setminus \{0\}$ and $p, a \in \mathbb{Z}$, then $h\left(\frac{p}{q}\right) = h\left(\frac{a}{b}\right)$.

(ii) Show that h satisfies properties (i)-(iv) from question 5 for all $m, n \in \mathbb{Q}$.

The function h is an ordered field isomorphism from $\mathbb{Q}$ to $\mathbb{Q}' \subseteq F$. This shows that any ordered field F has a subfield $\mathbb{Q}'$ that is isomorphic to $\mathbb{Q}$.