Homework 1 Solutions

CS 117, S25

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(a)

By part (v) of Thm from class, $0<\frac{1}{b}$, $0<\frac{1}{a}$, so it suffices to show $\frac{1}{b}<\frac{1}{a}$.

Note that $a<b \Rightarrow a \leq b$

$\overset{(O5)}{\Longrightarrow} a\cdot\left(\frac{1}{a}\cdot\frac{1}{b}\right) \leq b\cdot\left(\frac{1}{a}\cdot\frac{1}{b}\right) \overset{(M2)}{\Longrightarrow} a\cdot\left(\frac{1}{a}\cdot\frac{1}{b}\right) \leq b\cdot\left(\frac{1}{b}\cdot\frac{1}{a}\right)$

$\overset{(M1)}{\Longrightarrow} \left(a\cdot\frac{1}{a}\right)\cdot\frac{1}{b} \leq \left(b\cdot\frac{1}{b}\right)\cdot\frac{1}{a} \overset{(M4)}{\Longrightarrow} 1\cdot\frac{1}{b} \leq 1\cdot\frac{1}{a}$

$\overset{(M2)}{\Longrightarrow} \frac{1}{b}\cdot 1 \leq \frac{1}{a}\cdot 1 \overset{(M3)}{\Longrightarrow} \frac{1}{b} \leq \frac{1}{a}$.

It remains to show $\frac{1}{b} \neq \frac{1}{a}$. Suppose for the sake of contradiction that $\frac{1}{b}=\frac{1}{a}$.

Then, $\frac{1}{b}=\frac{1}{a} \Rightarrow \frac{1}{b}(a\cdot b)=\frac{1}{a}(a\cdot b)$

$\overset{(M1 \text{ and } M2)}{\Longrightarrow} \left(\frac{1}{b}\cdot b\right)\cdot a=\left(\frac{1}{a}\cdot a\right)\cdot b \overset{(M4)}{\Longrightarrow} 1\cdot a=1\cdot b \overset{(M2, M3)}{\Longrightarrow} a=b$,

which contradicts that $a<b$. Therefore $\frac{1}{b}\neq\frac{1}{a}$.

(b) By part (iv) of the theorem from class, $0 \leq (a+b)^2$. By (A1), (A2), (M1), (M2), (DL),

$$(a+b)\cdot(a+b) = a(a+b) + b(a+b)$$
$$= a^2 + ab + ab + b^2$$
$$= a^2 + ab(1+1) + b^2$$
$$= a^2 + 2ab + b^2;$$

so $0 \leq a^2 + 2ab + b^2$. Thus, by (A1), (A2), (A4), (O4), $-2ab \leq a^2 + 2ab + b^2 - 2ab = a^2 + b^2$.

(c) Assume, for the sake of contradiction, that $a > b$. By prop from class, $\exists\, r \in F$ s.t. $b < r < a$. By (A4), (O4), $r - b > 0$. Let $\varepsilon := r - b$. Then by (A1-4), $b + \varepsilon = b + r - b = r < a$, which contradicts our hypothesis that $a \leq b + \varepsilon$ for all $\varepsilon > 0$.

(4)

(a) Fix $x, y \in \mathbb{Q}(i)$, so $x = p + qi$ and $y = p' + q'i$ for some $p, p', q, q' \in \mathbb{Q}$. Since $\mathbb{Q}$ is an ordered field, either

(1) $p < p'$

(2) $p = p'$

(3) $p > p'$

In cases (1) and (3), we have $x < y$ and $x > y$, respectively. In case (2), since $\mathbb{Q}$ is an ordered field, either

(A) $q < q'$

(B) $q = q'$

(C) $q > q'$.

In cases (A) and (C), we have $x < y$ and $x > y$, respectively. In case (B), we have $x = y$.

(b) Let $x = p + qi$, $y = p' + q'i$, and $z = p'' + q''i$ for $p, p', p'', q, q', q'' \in \mathbb{Q}$. Suppose $x \leq y$ and $y \leq z$. Then, there are four possible cases:

(1) $p < p'$, $p' < p''$,
(2) $p < p'$, $p' = p''$ and $q' \le q''$,
(3) $p = p'$, $q \le q'$, and $p' < p''$,
(4) $p = p'$, $q \le q'$, $p' = p''$ and $q' \le q''$.

The result now follows since $\mathbb{Q}$ is an ordered field.
In cases (1-3), we have $p < p''$, so $x \le z$.
In case (4), we have $p = p''$ and $q \le q''$, so $x \le z$.

(c) $\mathbb{Q}(i)$ fails property (O5).

Note that $i = 0 + 1i \ge 0 + 0i = 0$.
However $0 \cdot i \not\le i \cdot i$, since

$$i \cdot i = (0+1i)\cdot(0+1i) = -1 = -1 + 0i < 0 + 0i.$$

(6)

(a) Note that $|b| \le a$ ensures $a \ge 0$. Then,

$$\begin{aligned} |b| \le a \iff & \begin{cases} b \le a & \text{if } b \ge 0 \\ -b \le a & \text{if } b \le 0 \end{cases} \\ \iff & \begin{cases} -a \le 0 \le b \le a & \text{if } b \ge 0 \\ -a \le b \le 0 \le a & \text{if } b \le 0 \end{cases} \\ \iff & -a \le b \le a. \end{aligned}$$

(b) By the triangle inequality, for all $a, b \in F$,

$$|b| = |a + (b-a)| \leq |a| + |a-b|.$$

Thus,

$$(*) \quad |b| - |a| \leq |a-b|.$$

Since $a, b \in F$ were arbitrary, we also have

$$(**) \quad |a| - |b| \leq |b-a| = |a-b| \implies -|a-b| \leq |b| - |a|.$$

Combining $(*)$ and $(**)$, by part (a),

$$\big||b| - |a|\big| \leq |a-b|.$$

(7)

(a) By (6)(a),

$$|a-b| \leq c \iff -c \leq a-b \leq c \overset{(O4)}{\iff} b-c \leq a \leq b+c.$$

(b) In view of part (a), it suffices to show

$$|a-b| \neq c \iff b-c \neq a \text{ and } a \neq b+c,$$

for $c \geq 0$ arbitrary.

To see this, note that

$$\begin{aligned} |a-b| \neq c \iff\ & a-b \neq c && \text{if } a > b \\ & b-a \neq c && \text{if } a < b \\ \iff\ & a \neq b+c && \text{if } a > b \\ & a \neq b-c && \text{if } a < b \end{aligned}$$

$\Leftrightarrow \quad \underbrace{a \neq b+c}_{\text{since } c>0 \text{, this is only possible if } a>b} \quad \text{and} \quad \underbrace{a \neq b-c}_{\text{since } c>0 \text{, this is only possible if } a<b.}$

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[insert]

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(i) By definition, f is surjective. If $f(n)=f(m)$, then $\sum_{i=1}^{n} \underline{1}' = \sum_{i=1}^{m} \underline{1}'$ Q1(a)

WLOG $n \leq m$

$\Rightarrow$ $\underbrace{\underline{1}'+\underline{1}'+\ldots+\underline{1}'}_{m-n \text{ times}} = (m-n)\underline{1}'=0 \Rightarrow m=n.$

Thus, f is injective.

(ii) This follows from associativity and commutativity of addition.

(iii) Recall that $0'<\underline{1}'$, so by (O4), $0<\underbrace{\underline{1}'+\ldots+\underline{1}'}_{a \text{ times}}$ for any $a \in \mathbb{N}$. Therefore,

$$
\begin{aligned}
n<m &\Leftrightarrow m-n>0 \\
&\Leftrightarrow \underbrace{\underline{1}'+\underline{1}'+\ldots+\underline{1}'}_{m-n \text{ times}}>0 \\
&\Leftrightarrow m'-n'>0 \\
&\Leftrightarrow m'>n'.
\end{aligned}
$$

(iv) $f(nm) = \sum_{i=1}^{nm} \underline{1}' \overset{(D2)}{=} \left(\sum_{i=1}^{n} \underline{1}'\right)\left(\sum_{i=1}^{m} \underline{1}'\right) = f(n)f(m).$

(9)

Let $g(a) = \begin{cases} f(a) & \text{if } a \in \mathbb{N} \\ 0' & \text{if } a = 0 \\ -f(-a) & \text{if } -a \in \mathbb{N}. \end{cases}$

In the previous solution, we showed that $\forall\, a \in \mathbb{N}'$, $a > 0$. Thus $\forall\, b \in -\mathbb{N}'$, $b < 0$.

Since $f: \mathbb{N} \to \mathbb{N}'$ is surjective, g is surjective. If $g(a) = g(b)$, then either

(A) $g(a) = g(b) > 0' \Rightarrow g(a) = f(a) = f(b) = g(b)$

$\Rightarrow a = b$, by injectivity of f

(B) $g(a) = g(b) < 0' \Rightarrow f(-a) = f(-b)$

$\Rightarrow -a = -b$, by injectivity of f

$\Rightarrow a = b$

(C) $g(a) = g(b) = 0 \Rightarrow a = b = 0$, by defn of g.

Thus g is injective.

To see (ii), note that $\forall\, a, b \in \mathbb{Z}$,

Case 1: either $a = 0$ or $b = 0$. WLOG, suppose $b = 0$, so

$$g(a) + g(b) = g(a) + 0' = g(a + 0)$$

Case 2: $a > 0$ and $b > 0$. The result follows from Q8.

Case 3: $a < 0$ and $b < 0$.

$g(a)+g(b) = -f(-a)-f(-b) = -f(-a-b) = g(a+b)$

Case 4: a and b have opposite signs. WLOG $a>0, b<0$.

Then, $g(a)+g(b) = f(a)-f(-b)$.

If $a>-b$, let $m=a+b$, $n=-b$, so

$$f(a)-f(-b) = f(m+n)-f(n) \overset{Q8(ii)}{=} f(m) = g(a+b).$$

If $-b>a$, let $m=-b-a$, $n=a$, so

$$f(a)-f(-b) = f(n)-f(m+n) \overset{Q8(ii)}{=} -f(m) = g(a+b).$$

This shows g satisfies (ii).

To see (iii), note that

$$
\begin{aligned}
a<b &\iff b-a>0 \\
&\overset{(ii)}{\iff} g(b)+g(-a) = g(b-a) = f(b-a) > 0 \\
&\overset{g(a)=-g(-a) \text{ by def of } g}{\iff} g(b)-g(a) > 0 \\
&\iff g(a)<g(b).
\end{aligned}
$$

Finally, to see (iv), for $a,b \in \mathbb{Z}$,

Case 1: either $a=0$ or $b=0$. By (ii), we see $g(0)=0'$;

so $g(ab) = g(0) = 0' = g(a)g(b)$.

Case 2: a and b have same sign

Then $g(ab) = f(ab) = \begin{cases} f(a)f(b) & \text{if } a,b>0 \\ f(-a)f(-b) & \text{if } a,b<0 \end{cases}$

$= g(a)g(b)$

Case 3: a and b have opposite signs. WLOG $a>0$.

Then $g(ab) = -f(a(-b)) = -f(a)f(-b) = g(a)g(b)$.