Math 117: Homework 2

Due Monday, April 14th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1*

The following theorem provides a mild extension of the principle of mathematical induction.

THEOREM 1. *Fix $m \in \mathbb{N}$. Given a list of propositions*

$$\{P_m, P_{m+1}, P_{m+2}, \dots\} = \{P_k : k \in \mathbb{N}, k \geq m\},$$

suppose the following holds:

(i) P_m is true;

(ii) for all $n \in \mathbb{N}$ with $n \geq m$, if P_n is true, then P_{n+1} is true;

Then $P_m, P_{m+1}, P_{m+2} \dots$ are all true.

Prove the theorem.

Question 2*

Use Question 1 to prove the following statements:

(a) $n^2 > n + 1$ for all natural numbers $n \geq 2$.

(b) $n! > n^2$ for all natural numbers $n \geq 4$. (Recall that $n! = n(n-1) \cdot \dots \cdot 2 \cdot 1$. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.)

Question 3*

Consider an ordered field F and fix $p, q \in F$. Suppose that for all $s \in F$ with $s > p$, we have $q \leq s$. Prove that $q \leq p$.

Question 4

Consider an ordered field F and suppose $S \subseteq F$ is nonempty and bounded above. Prove that a is the supremum of S if and only if a is an upper bound for S and, for all $\epsilon > 0$, there exists $s \in S$ so that $s > a - \epsilon$.

Question 5

Consider an ordered field F and a nonempty subset $S \subseteq F$.

(a) If the maximum of S exists, prove that the maximum is the supremum of S.

(b) If $\sup S \in S$, prove that $\max(S) = \sup(S)$.

(c) If $\sup S \notin S$, prove that $\max(S)$ does not exist.

Question 6

Let S be a nonempty bounded subset of $\mathbb{R}$.

(a) Prove that $\inf S \leq \sup S$.

(b) What can you say about the number of elements in S if $\inf S = \sup S$? Justify your answer with a proof.

Question 7*

Suppose S and T are nonempty bounded subsets of $\mathbb{R}$.

(a) Prove that if $S \subseteq T$, then $\inf T \leq \inf S \leq \sup S \leq \sup T$.

(b) Prove $\sup(S \cup T) = \max\{\sup S, \sup T\}$. (Note: for this part, do not assume $S \subseteq T$.)

Question 8*

Consider the following proposition:

PROPOSITION 2. *Every nonempty subset S of $\mathbb{R}$ that is bounded below has an infimum.*

This question will lead you through the proof of the proposition.

(a) Suppose S is as in the proposition above. Define the set $-S = \{-s : s \in S\}$. Show that $-S$ is bounded above.

(b) Use the definition of $\mathbb{R}$ to prove that $-S$ has a supremum, $\sup(-S)$.

(c) Prove that $-\sup(-S)$ is the infimum of S.

Question 9

Let S and T be nonempty bounded subsets of $\mathbb{R}$, and define $S + T = \{s + t : s \in S \text{ and } t \in T\}$. Prove $\sup(S + T) = \sup S + \sup T$.

Question 10*

For each of the sets below, answer the following questions: Is it bounded above? If so, what is its supremum? Is it bounded below? If so, what is its infimum? You do not need to justify your answers.

(a) $[-\sqrt{2}, \sqrt{2}]$

(b) $\{-1, 0, e, \pi\}$

(c) $\{1\}$

(d) $\cup_{n=1}^{\infty}[2n-1, 2n)$

(e) $\left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$

(f) $\{x \in \mathbb{R} : x^2 < 1\}$

(g) $\cap_{n=1}^{\infty}(-1 - \frac{1}{n}, 1 + \frac{1}{n})$

Question 11*

Where is the flaw in the following "proof" by induction that any two positive integers are equal?

THEOREM 3. *Consider* $k, m \in \mathbb{N}$*. If* $n = \max\{k, m\}$*, then* $n = k = m$*.*

Proof. Let $S(n)$ be the statement of the theorem. If $1 = \max\{k, m\}$, then $k = m = 1$ by definition of $\mathbb{N}$. Suppose $n+1 = \max\{k, m\}$. Then $n = \max\{k-1, m-1\}$, so by induction, $n = k-1 = m-1$, and thus $n + 1 = k = m$. □

Question 12

Suppose $X \subseteq \mathbb{N}$ is nonempty and bounded above. Prove that $\max(X)$ exists.