Homework 2 Solutions

CS117

① Define the propositions $\{Q_1, Q_2, \dots\} = \{Q_\ell : \ell \in \mathbb{N}\}$ by

$$Q_\ell := P_{\ell-1+m}$$

Then Q_1 is true and for all $\ell \in \mathbb{N}$, $\ell-1+m \geq m$, so if $Q_\ell = P_{\ell-1+m}$ is true, then $Q_{\ell+1} = P_{(\ell-1+m)+1}$ is true. Therefore, by induction

$$\{Q_1, Q_2, \dots\} = \{P_m, P_{m+1}, \dots\}$$

are all true.

② ⓐ Base case: When $n=2$, we have $2^2=4>3=2+1$.
Inductive step: Suppose $n^2>n+1$. Then
$(n+1)^2=n^2+2n+1>(n+1)+2n+1\geq(n+1)+1$,
which completes the proof.
ⓑ Base case: When $n=4$, we have $4!=24>16=4^2$.
Inductive step: Suppose $n!>n^2$. Then
$(n+1)!=(n+1)\cdot n!>(n+1)n^2$. By part (a),
$n^2>n+1$ for all $n\geq 2$. Therefore, we
may continue the previous inequality to
obtain $(n+1)n^2>(n+1)^2$. Hence $(n+1)!>(n+1)^2$,
which completes the proof.

③ Assume for the sake of contradiction that $q > p$. By the proposition from class, there exists $r \in F$ so that $q > r > p$. By assumption, since $r > p$, we must have $q \leq r$. This contradicts the fact that $q > r$. Therefore, we must have $q \leq p$.

(5)

(a) Let $M = \max(S)$. By definition of maximum, $s \le M$ for all $s \in S$, so M is an upper bound of S. If $\tilde{M}$ is an upper bound of S, then since the defition of maximum ensures $M \in S$, we have $\tilde{M} \ge M$. Thus, M is the least upper bound of S, so $M = \sup(S)$.

(b) By definition, $\sup(S) \ge s$ for all $s \in S$. Thus, if $\sup(S) \in S$, it is the largest element of S and $\max(S) = \sup(S)$.

(c) Suppose S has a maximum M_0. By (a), $M_0 = \sup(S)$. This contradicts the fact that $\sup(S) \notin S$. Therefore, S must not have a maximum.

(6) (a) Let s be an element of S. Since $\inf(S)$ is a lower bound for S, $\inf(S) \leq s$. Since $\sup(S)$ is an upper bound for S, $s \leq \sup(S)$. Therefore, $\inf(S) \leq \sup(S)$.

(b) We will show $S = \{\inf(S)\}$, so there is one element in the set. Since $\inf(S) = \sup(S)$, $\inf(S)$ is both an upper and lower bound for S. In particular, for any $s \in S$, $\inf(S) \leq s$ and $\inf(S) \geq s$. Thus $\inf(S) = s$ for all $s \in S$. This shows $S = \{\inf(S)\}$.

7 (a) Since $\inf(T)$ is a lower bound for the set T, if $S \subseteq T$, then $\inf(T)$ is also a lower bound for the set S. Since $\inf(S)$ is the greatest lower bound of S, $\inf(T) \leq \inf(S)$. The fact that $\sup(S) \leq \sup(T)$ follows from an analogous argument. The fact that $\inf(S) \leq \sup(S)$ follows from Q6.

you should write this argument or use $-S$ and $-T$ to reduce to what you have already shown

(b) Since $\sup(S)$ is an upper bound for S and $\sup(T)$ is an upper bound for T, $\max\{\sup(S), \sup(T)\}$ is an upper bound for $S \cup T$. Thus, since $\sup(S \cup T)$ is the least upper bound for $S \cup T$, $\sup(S \cup T) \leq \max\{\sup(S), \sup(T)\}$.

By Q7 (a), since $S \subseteq S \cup T$ and $T \subseteq S \cup T$, we have $\sup(S) \leq \sup(S \cup T)$ and $\sup(T) \leq \sup(S \cup T)$. Thus $\max\{\sup(S), \sup(T)\} \leq \sup(S \cup T)$.

Combining these two inequalities, we conclude $\max\{\sup(S), \sup(T)\} = \sup(S \cup T)$.

(8) (a) Since S is bounded below, $\exists\, m_0 \in \mathbb{R}$ s.t. $s \geq m_0 \ \forall s \in S$. This implies $-m_0 \geq -s \ \forall s \in S$, so $-S$ is bounded above.

(b) Since S is nonempty, so is $-S$. Since $-S$ is bounded above, by definition of the real numbers, it has a supremum, $\sup(-S)$.

(c) Since $\sup(-S)$ is an upper bound for $-S$, $-s \leq \sup(-S)$ for all $s \in S$, hence $s \geq -\sup(-S)$ for all $s \in S$. Therefore $-\sup(-S)$ is a lower bound for S, and it suffices to show it is the greatest lower bound.

Suppose for the sake of contradiction that m_0 is a lower bound for S with $m_0 > -\sup(-S)$. As argued in part (a), $-m_0$ is an upper bound for $-S$. Furthermore, $m_0 > -\sup(-S)$ implies $-m_0 < \sup(-S)$. This is a contradiction, since $\sup(-S)$ is the least upper bound for $-S$.

Therefore $-\sup(-S)$ must be the greatest lower bound of $-S$.

(9) Step 1: Show that for all $t \in T$, $\inf(S+T)-t$ is a lower bound for S.

By defn of $S+T$ and the infimum, $\inf(S+T)$ is a lower bound for $S+T$, so $s+t \geq \inf(S+T) \iff s \geq \inf(S+T)-t$ for all $s \in S$, $t \in T$. Thus, for all $t \in T$, $\inf(S+T)-t$ is a lower bound for S.

Step 2: Show that $\inf(S+T)-\inf(S)$ is a lower bound for T.

By Step 1, for all $t \in T$, $\inf(S+T)-t$ is a lower bound for S. By defn, $\inf(S)$ is the greatest lower bound of S. Thus, $\inf(S) \geq \inf(S+T)-t$

$\iff t \geq \inf(S+T)-\inf(S)$ for all $t \in T$.

Since $\inf(S+T) - \inf(S)$ is a lower bound for T and $\inf(T)$ is the greatest lower bound,

$$\inf(T) \geq \inf(S+T) - \inf(S).$$
$$\Updownarrow$$
$$\inf(S) + \inf(T) \geq \inf(S+T). \quad (*)$$

It remains to prove the opposite inequality. Since $\inf(S)$ and $\inf(T)$ are lower bounds for S and T, for all $s \in S$ and $t \in T$, $\inf(S) \leq s$ and $\inf(T) \leq t \Rightarrow \inf(S) + \inf(T) \leq s+t$. Thus, $\inf(S) + \inf(T)$ is a lower bound for $S+T$. Since $\inf(S+T)$ is the greatest lower bound,

$$\inf(S) + \inf(T) \leq \inf(S+T). \quad (**)$$

Thus, combining inequalities $(*)$ and $(**)$, we obtain

$$\inf(S) + \inf(T) = \inf(S+T). \quad \square$$

(10) Throughout, we use S to denote the set under consideration.

(a) $\sup(S) = \sqrt{2}$, $\inf(S) = -\sqrt{2}$

(b) $\sup(S) = \pi$, $\inf(S) = -1$

(c) $\sup(S) = \inf(S) = 1$

(d) S is not bounded above, $\inf(S) = 1$

(e) $\sup(S) = 1$, $\inf(S) = 0$

(f) $\sup(S) = 1$; $\inf(S) = -1$

(g) $S = [-1, 1]$, so $\sup(S) = 1$ and $\inf(S) = -1$

(11) The flaw occurs in the final sentence. The fact that $k, m \in \mathbb{N}$ does not guarantee $k-1, m-1 \in \mathbb{N}$, as was used in the base case. For example, take $k=2, m=1$.