Homework 3 Solutions

CS 117

① Fix $\varepsilon > 0$ arbitrary. By definition of convergence, $\exists$ N s.t. $n > N$ ensures $|s_n - s| < \varepsilon$ and $\exists$ $\tilde{N}$ s.t. $n > \tilde{N}$ ensures $|s_n - \tilde{s}| < \varepsilon$. Then $n > \max\{N, \tilde{N}\}$ ensures

$$|s - \tilde{s}| = |s - s_n + s_n - \tilde{s}| \overset{\Delta \text{ ineq}}{\leq} |s - s_n| + |s_n - \tilde{s}| \leq 2\varepsilon$$

Since $\varepsilon > 0$ was arbitrary, this shows $s = \tilde{s}$.

(2)

(a) We estimate as follows:

$$(x+\varepsilon)^2 = x^2+2\varepsilon x+\varepsilon^2 \overset{x\le 2}{\le} x^2+4\varepsilon+\varepsilon^2 \overset{\varepsilon\le 1 \Rightarrow \varepsilon^2\le\varepsilon}{\le} x^2+5\varepsilon$$

$$(y-\varepsilon)^2 = y^2-2\varepsilon y+\varepsilon^2 \overset{y\le 2}{\ge} y^2-4\varepsilon+\varepsilon^2 \overset{\varepsilon^2\ge 0}{\ge} y^2-4\varepsilon.$$

(b) Since $x^2<2$, $\tilde{\varepsilon}_1 := 2-x^2>0$. Let $\varepsilon_1 = \frac{\tilde{\varepsilon}_1}{10}$.
Then $x^2+5\varepsilon_1 = x^2+\frac{\tilde{\varepsilon}_1}{2} < x^2+\tilde{\varepsilon}_1 = 2$.

Since $y^2>2$, $\tilde{\varepsilon}_2 = y^2-2>0$. Let $\varepsilon_2 = \frac{\tilde{\varepsilon}_2}{8}$.
Then $y^2-4\varepsilon_2 = y^2-\frac{\tilde{\varepsilon}_2}{2} > y^2-\tilde{\varepsilon}_2 = 2$.

(c) We estimate as follows:

$$(x+\varepsilon_1)^2 \overset{(a)}{\le} x^2+5\varepsilon_1 \overset{(b)}{<} 2$$

$$(y-\varepsilon_2)^2 \overset{(a)}{\ge} y^2-4\varepsilon_2 \overset{(b)}{>} 2.$$

(3)

(a) As shown in class, $0^2 = 0$. Thus $0 \in S$, so $S \neq \emptyset$.

Define $m = \max\{a, 1\}$. Suppose $c \in S$. If $c \leq 1$, then $c \leq 1 \leq m$. If $c \geq 1$, then by (O5), $c^2 \geq c$, so $c \leq c^2 \leq a \leq m$. This shows S is bounded above by m. By definition, S is bounded below by 0.

(b) By definition of $\mathbb{R}$, for any nonempty subset of $\mathbb{R}$ that is bounded above, the supremum exists.

4

a) In 3 a, we showed $m = \max\{2,1\} = 2$ is an upper bound for S. Thus $b \le 2$. Since $1 \ge 0$ and $1^2 \le 2$, $1 \in S$, so $b \ge 1$.

b) If $b^2 < 2$, by 2 c $\exists\, \varepsilon_1 \in (0,1)$ s.t. $(b+\varepsilon_1)^2 < 2$. Thus $b + \varepsilon_1 \in S$. This contradicts the fact that b is an upper bound of S.

c) If $b^2 > 2$, by 2 c, $\exists\, \varepsilon_2 \in (0,1)$ s.t. $(b-\varepsilon_2)^2 > 2$. Thus, if $c > b - \varepsilon_2$,

$$c^2 \overset{c \ge 0}{\ge} c(b-\varepsilon_2) \overset{b-\varepsilon_2 \ge 0}{\ge} (b-\varepsilon_2)^2 > 2,$$

so $c \notin S$. By Q1, this contradicts that b is the supremum.

(5) We first show $\lim a^n = 0$ if $|a|<1$.

(a) Note that, if $a=0$, then for all $\varepsilon>0$ and any $N\in\mathbb{R}$, $n>N$ ensures $|a^n-0|=0<\varepsilon$. Thus, $\lim a^n=0$.

Now, suppose that $a\neq 0$.
Fix $\varepsilon>0$. Note that $|a^n-0|<\varepsilon$
$\Leftrightarrow |a^n|<\varepsilon \Leftrightarrow |a|^n<\varepsilon \Leftrightarrow n\log(|a|)<\log\varepsilon$

[since $|a^n|=|a|^n$

since $|a|<1$ ensures $\log(|a|)<0$

$\Leftrightarrow n>\frac{\log\varepsilon}{\log(|a|)}$. Let $N=\frac{\log\varepsilon}{\log(|a|)}$.
Then $n>N$ ensures $|a^n-0|<\varepsilon$, so $\lim a^n=0$.

We now show $\lim a^n=1$ if $a=1$.
Note that if $a=1$, then $a^n=1$ for all n.
Fix $\varepsilon>0$ and choose $N=1$. Then $n>N$ ensures $|a^n-1|=0<\varepsilon$.

(b) We conclude by showing a^n does not converge if $a\leq -1$.

Suppose for the sake of contradiction that a^n converges to some $a \in \mathbb{R}$.
Let $\varepsilon = 1$. Then there exists N s.t. $n > N$ ensures $|a^n - a| < 1 \iff a-1 < a^n < a+1$.
For n even, $a^n = |a|^n > 1$ so $1 < a+1 \Rightarrow 0 < a$.
For n odd, $a^n = -|a|^n < -1$, so $a-1 < -1 \Rightarrow a < 0$.
This is a contradiction, since no $a \in \mathbb{R}$ can satisfy both $a > 0$ and $a < 0$.

6

(a) We will show that the minimum of S exists if $a \in \mathbb{Q}$ and does not exist if $a \in \mathbb{R} \setminus \mathbb{Q}$.

Suppose $a \in \mathbb{Q}$. By definition of S, $a \in S$. Furthermore, for any $s \in S$, $a \leq S$. Thus $\min(S) = a$.

Suppose $a \in \mathbb{R} \setminus \mathbb{Q}$. Assume, for the sake of contradiction, that the minimum of S exists, and $\min(S) = s_0$. Since $s_0 \in S$, we have $s_0 \geq a$. However, since $a \notin \mathbb{Q}$, we must have $s_0 > a$. By density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $r \in \mathbb{Q}$ so that $s_0 > r > a$. By definition of S, we must have $r \in S$. This contradicts that s_0 was the minimum of S. Thus, the minimum of S must not exist.

(b) We will show that $\inf(S) = a$. By definition of S, $a \leq s$ for all $s \in S$, so a is a lower bound for S. Suppose $m_0 \in \mathbb{R}$ is another lower bound of S. Assume for the sake of contradiction that $m_0 > a$. By density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $r \in \mathbb{Q}$ so that $m_0 > r > a$. Thus, $r \in S$, which contradicts the definition of m_0 as a lower bound of S. This shows $m_0 \leq a$, so a is the greatest lower bound.

⑦ Let $S = (a, b]$.

- $\max(S) = b$, since by defn, b is the largest element in S
- $\sup(S) = b$. b is an upper bound for S and since $b \in S$, no number smaller than b can be an upper bound. Thus b is the least upper bound.
- The minimum of S does not exist. Suppose, for the sake of contradiction that $\min(S) = m_0$. Since $m_0 \in S$, $m_0 > a$. However $\frac{m_0+a}{2} \in (a, m_0)$, so $\frac{m_0+a}{2} \in S$ and $\frac{m_0+a}{2} < m_0$. This contradicts that m_0 was the smallest element in S.
- $\inf(S) = a$. a is a lower bound for S. Suppose $m_0 > a$ was another lower bound. Since $b \in S$, we have $m_0 \in (a, b]$. Furthermore, since $\frac{m_0+a}{2} \in (a, m_0)$ we have $\frac{m_0+a}{2} \in S$ and $\frac{m_0+a}{2} < m_0$.

This contradicts that m_0 was a lower bound of S.

⑧ Suppose s_n is a bounded sequence. Then $\exists\ M > 0$ s.t. $|s_n| \leq M\ \forall n \in \mathbb{N}$
$\iff -M \leq s_n \leq M\ \ \forall n \in \mathbb{N}$.
Thus $-M$ is a lower bound for S and M is an upper bound for S, so S is bounded.

Now suppose S is bounded, so that it has lower and upper bounds m_0 and m_1. Define $M := \max\{|m_0|, |m_1|\}$. Then, $\forall n \in \mathbb{N}$,

$$-M \leq -|m_0| \leq m_0 \leq s_n \leq m_1 \leq |m_1| \leq M,$$

so $|s_n| \leq M$ $\forall n \in \mathbb{N}$ and s_n is a bounded sequence.

9

(a) A sequence s_n converges to a limit $s \in \mathbb{R}$ if, $\forall \varepsilon > 0$, $\exists N \in \mathbb{R}$ s.t. $n > N$ ensures $|s_n - s| < \varepsilon$.

(b) A sequence s_n does not converge to a limit $s \in \mathbb{R}$ if $\exists$ $\varepsilon > 0$ s.t. for all $N \in \mathbb{R}$, $\exists$ $n > N$ s.t. $|s_n - s| \geq \varepsilon$.

(c) Fix $\varepsilon>0$. Let $N=\frac{4}{\varepsilon}$. Then $n>N$ ensures

$$\frac{4}{n}<\varepsilon \iff \frac{4n}{n^2}<\varepsilon \iff \frac{n+3n}{n^2}<\varepsilon \implies \frac{n+3}{n^2}<\varepsilon \implies$$

$$\frac{|n-3|}{n^2}<\varepsilon \implies \left|\frac{n-3}{n^2}\right|<\varepsilon \implies \left|\frac{n-3}{n^2+9}\right|<\varepsilon \iff \left|\frac{n-3}{n^2+9}-0\right|<\varepsilon.$$

Since $\varepsilon>0$ was arbitrary, this gives the result.

(d) Assume, for the sake of contradiction, that s_n converges to some $s\in\mathbb{R}$. Then, for $\varepsilon=1$, there exists $N\in\mathbb{R}$ so that $n>N$ ensures

$$|s_n-s|<1 \iff s-1<s_n<s+1$$
$$\iff s-1<(n+1)^2-2<s+1$$
$$\iff s+1<(n+1)^2<s+3$$

By the lemma following the Archimedean Property, $\exists\, m\in\mathbb{N}$ so that $m>s+3$. Let $k=\max(m,N+1)$. Then $\underline{k\geq m>s+3}$ and $k>N$. The latter ensures:

$(k+1)^2 < s+3 \Rightarrow k < k^2+2k+1 < s+3$

This contradicts ⊛. Thus s_n must not converge to any $s \in \mathbb{R}$.

10

We will show a_n converges to $a = 7/3$. Fix $\epsilon > 0$. Note that

$$\begin{aligned}|a_n - a| &= \left|\frac{7n-19}{3n+7} - \frac{7}{3}\right| = \left|\frac{21n-57}{3(3n+7)} - \frac{7(3n+7)}{3(3n+7)}\right| \\ &= \left|\frac{-106}{3(3n+7)}\right| < \epsilon\end{aligned}$$

if and only if

$$\frac{108}{3\epsilon} < 3n+7 \Longleftarrow \frac{108}{3\epsilon} < 3n \iff \frac{108}{\epsilon} < n.$$

Thus, if $N = \frac{108}{\epsilon}$, for all $n > N$, we have $|a_n - a| < \epsilon$. Since ϵ was arbitrary, this shows $\lim_{n\to+\infty} a_n = a$.

We will show b_n does not converge. Note that the elements in the sequence b_n are $(\frac{1}{2}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2}, 0, \dots)$, repeating in this way. Assume, for the sake of contradiction, that b_n converges to some $b \in \mathbb{R}$. Then for $\epsilon = \frac{1}{4} > 0$, there exists N so that $n > N$ ensures

$$|b_n - b| < \epsilon \iff b - \epsilon < b_n < b + \epsilon \iff b - \frac{1}{4} < b_n < b + \frac{1}{4}.$$

Since there are infinitely many $n > N$ for which $b_n = -\frac{1}{2}$, we see that

$$b - \frac{1}{4} < -\frac{1}{2} \implies b < \frac{-1}{4}.$$

Likewise, since there are infinitely many $n > N$ for which $b_n = \frac{1}{2}$, we see that

$$\frac{1}{2} < b + \frac{1}{4} \implies \frac{1}{4} < b.$$

It is impossible to have both $b < \frac{-1}{4}$ and $\frac{1}{4} < b$. Thus, we have found a contradiction. This shows b_n does not converge.

(11) Fix $\varepsilon > 0$. Since a_n converges to a, $\exists\ N_0$ s.t. $n > N_0$ ensures $|a_n - a| < \varepsilon$. Choose $N_1 := \max\{N_0, N\}$. Then $n > N_1$ ensures $|b_N - a| = |a_n - a| < \varepsilon$. Thus, b_N converges to a.

(8)

(a) A sequence s_n does not converge if, for all $s \in \mathbb{R}$, $\exists\ \varepsilon > 0$ s.t. $\forall N \in \mathbb{R}$ $\exists\ n > N$ s.t. $|s_n - s| > \varepsilon$.

(b) (i) Fix $\varepsilon > 0$. Note that

$$|s_n - 0| = \left|\frac{1}{n}\sin(2n)\right| = \frac{1}{n}|\sin(2n)| \leq \frac{1}{n} < \varepsilon$$

if $n > \frac{1}{\varepsilon}$. Thus, for $N = \frac{1}{\varepsilon}$, we have that, for all $n > N$, $|s_n - 0| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this shows $\lim s_n = 0$.

(ii) Fix $\varepsilon > 0$. Note that

$$|d_n - \tfrac{3}{5}| = \left|\frac{3n+4}{5n+3} - \frac{3}{5}\right| = \left|\frac{(3n+4)5 - 3(5n+3)}{5(5n+3)}\right|$$

$$= \left|\frac{11}{5(5n+3)}\right| = \frac{11}{5}\,\frac{1}{5n+3} \leq \frac{11}{25}\cdot\frac{1}{n} \leq \frac{1}{2n} < \varepsilon$$

if $n > \frac{1}{2\varepsilon}$. Thus, for $N = \frac{1}{2\varepsilon}$, we have that, for all $n > N$, $|d_n - \frac{3}{5}| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this shows $\lim d_n = \frac{3}{5}$.

(iii) Fix $s \in \mathbb{R}$ arbitrary. Let $\varepsilon = 2$. Suppose $\exists\ N \in \mathbb{R}$ s.t. $\forall n > N$, $|a_n - s| < \varepsilon$.

Note that

$$|a_n - s| = |2\cos(n\pi) - s| < \varepsilon \iff$$
$$s - \varepsilon < 2\cos(n\pi) < s + \varepsilon \iff$$
$$s - 2 < 2\cos(n\pi) < s + 2.$$

In particular, for $n > N$ even, we have $2 < s + 2 \implies s > 0$. Similarly, for $n > N$ odd, we have $s - 2 < -2 \implies s < 0$. Since it is impossible to have both $s > 0$ and $s < 0$, we obtain a contradiction.

Thus $\forall N \in \mathbb{R}$, $\exists n > N$ s.t. $|a_n - s| \geq \varepsilon$. This shows a_n does not conver .

(10)

(a) A sequence s_n converges to a limit $s \in \mathbb{R}$ if, $\forall \varepsilon > 0$, $\exists N \in \mathbb{R}$ s.t. $n > N$ ensures $|s_n - s| < \varepsilon$.

(b) A sequence s_n does not converge to a limit $s \in \mathbb{R}$ if $\exists \varepsilon > 0$ s.t. for all $N \in \mathbb{R}$, $\exists n > N$ s.t. $|s_n - s| \geq \varepsilon$.

(c) Fix $\varepsilon > 0$. Let $N = \frac{4}{\varepsilon}$. Then $n > N$ ensures

$$\frac{4}{n} < \varepsilon \iff \frac{4n}{n^2} < \varepsilon \iff \frac{n+3n}{n^2} < \varepsilon \Longrightarrow \frac{n+3}{n^2} < \varepsilon \Longrightarrow$$

$$\frac{|n-3|}{n^2} < \varepsilon \Longrightarrow \left|\frac{n-3}{n^2}\right| < \varepsilon \Longrightarrow \left|\frac{n-3}{n^2+9}\right| < \varepsilon \iff \left|\frac{n-3}{n^2+9} - 0\right| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this gives the result.

(2) Assume, for the sake of contradiction, that s_n converges to some $s \in \mathbb{R}$. Then, for $\varepsilon = 1$, there exists $N \in \mathbb{R}$ so that $n > N$ ensures

$$|s_n - s| < 1 \iff s - 1 < s_n < s + 1$$
$$\iff s - 1 < (n+1)^2 - 2 < s + 1$$
$$\iff s + 1 < (n+1)^2 < s + 3$$

By the lemma following the Archimedean Property, $\exists\, m \in \mathbb{N}$ so that $m > s+3$. Let $k = \max(m, N+1)$. Then $\underline{k \geq m > s+3}$ and $k > N$. The latter ensures:

$$(k+1)^2 < s+3 \Longrightarrow k^2 + 2k + 1 < s+3$$

$(k+1)^2 < s+3 \Rightarrow k < k^2+2k+1 < s+3$

This contradicts ⊛. Thus s_n must not converge to any $s \in \mathbb{R}$.

We will show a_n converges to $a = 7/3$. Fix $\epsilon > 0$. Note that

$$|a_n - a| = \left|\frac{7n-19}{3n+7} - \frac{7}{3}\right| = \left|\frac{21n-57}{3(3n+7)} - \frac{7(3n+7)}{3(3n+7)}\right|$$
$$= \left|\frac{-106}{3(3n+7)}\right| < \epsilon$$

if and only if

$$\frac{108}{3\epsilon} < 3n+7 \Longleftarrow \frac{108}{3\epsilon} < 3n \iff \frac{108}{\epsilon} < n.$$

Thus, if $N = \frac{108}{\epsilon}$, for all $n > N$, we have $|a_n - a| < \epsilon$. Since ϵ was arbitrary, this shows $\lim_{n\to+\infty} a_n = a$.

We will show b_n does not converge. Note that the elements in the sequence b_n are $(\frac{1}{2}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2}, 0, \dots)$, repeating in this way. Assume, for the sake of contradiction, that b_n converges to some $b \in \mathbb{R}$. Then for $\epsilon = \frac{1}{4} > 0$, there exists N so that $n > N$ ensures

$$|b_n - b| < \epsilon \iff b - \epsilon < b_n < b + \epsilon \iff b - \frac{1}{4} < b_n < b + \frac{1}{4}.$$

Since there are infinitely many $n > N$ for which $b_n = -\frac{1}{2}$, we see that

$$b - \frac{1}{4} < -\frac{1}{2} \implies b < \frac{-1}{4}.$$

Likewise, since there are infinitely many $n > N$ for which $b_n = \frac{1}{2}$, we see that

$$\frac{1}{2} < b + \frac{1}{4} \implies \frac{1}{4} < b.$$

It is impossible to have both $b < \frac{-1}{4}$ and $\frac{1}{4} < b$. Thus, we have found a contradiction. This shows b_n does not converge.

11)

(a)

(b)