Math 117: Homework 4

Due Monday, May 5th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1

On Homework 3, Q5, you proved that, if $-1 < a < 1$, then $\lim_{n\to+\infty} a^n = 0$, and if $a = 1$, then $\lim_{n\to+\infty} a^n = 1$.

(a) If $a > 1$, prove that $\lim_{n\to+\infty} a^n = +\infty$.

(b) If $a \leq -1$, prove that the limit of a^n does not exist.

Question 2

Recall that the *reverse triangle inequality* ensures that, for all $a, b \in \mathbb{R}$, $||a| - |b|| \leq |a - b|$.

Suppose t_n is a convergent sequence, with $\lim_{n\to+\infty} t_n = t$.

(a) Use the definition of convergence to prove that

$$\lim_{n\to+\infty} |t_n| = |t|.$$

(b) As a consequence of part (a), you have proved the following statement:

If t_n is a convergent sequence, then $|t_n|$ is a convergent sequence.

Is the converse true? If so, prove it. If not, give a counterexample and justify your counterexample.

Question 3*

Prove the following theorem:

THEOREM 1. *If s_n and t_n are convergent sequences and $\lim_{n\to+\infty} s_n \neq 0$, then*

$$\lim_{n\to+\infty} \left(\frac{t_n}{s_n}\right) = \frac{\lim_{n\to+\infty} t_n}{\lim_{n\to+\infty} s_n}.$$

Question 4*

Prove the following useful lemma:

LEMMA 2. *If r_n and t_n are sequences whose limits exist and $r_n \leq t_n \ \forall n \in \mathbb{N}$, then*

$$\lim_{n\to+\infty} r_n \leq \lim_{n\to+\infty} t_n.$$

Do **not** assume that the sequences r_n and t_n converge, only that the limits exist.

Question 5*

An important lemma in the analysis of sequences is known as the *squeeze lemma.*

LEMMA 3 (Squeeze Lemma)**.** *Consider three sequences a_n, b_n, and s_n. If $a_n \leq s_n \leq b_n$ for all $n \in \mathbb{N}$ and*

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} b_n,$$

then $\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} a_n = \lim_{n\to+\infty} b_n$..

(a) Prove the squeeze lemma. (Hint: use Q4.)

(b) Suppose s_n and t_n are sequences satisfying $|s_n| \leq t_n$ for all $n \in \mathbb{N}$. If $\lim_{n\to+\infty} t_n = 0$, prove that $\lim_{n\to+\infty} s_n = 0$.

(c) Is the converse to part (b) true? If so, prove it. If not, give a counterexample and justify your counterexample.

Question 6

Suppose the limit of s_n exists and $k \in \mathbb{R}$. Then $\lim_{n\to+\infty} ks_n = k \lim_{n\to+\infty} s_n$, where we take the convention that

$$k \cdot (+\infty) = \begin{cases} +\infty & \text{if } k > 0 \\ 0 & \text{if } k = 0 \\ -\infty & \text{if } k < 0, \end{cases} \qquad \text{and} \qquad k \cdot (-\infty) = \begin{cases} -\infty & \text{if } k > 0 \\ 0 & \text{if } k = 0 \\ +\infty & \text{if } k < 0. \end{cases}$$

Question 7

Prove that, for any $a \in \mathbb{R}$, there exists an increasing sequence of rational numbers r_n so that $\lim_{n\to+\infty} r_n = a$.

Question 8*

State the converse to Practice Midterm Q3 and prove it.

Question 9*

For any $a \in \mathbb{R}$, $a > 0$, prove that $\lim_{n\to+\infty} a^{1/n} = 1$.

Hint: First prove the result for $a \geq 1$. In this case, begin by explaining why $a^{1/n}$ must converge to some $L \geq 1$. Then argue that $a^{2/n}$ must converge to $L^2 \geq 1$. Using Q8, explain why $L^2 = L$.