Homework 4 Solutions, CCS 117, S25

Question 1:

(a) If $a > 1$, prove that $\lim_{n\to+\infty} a^n = +\infty$.

(b) If $a < -1$, prove that the limit of a^n does not exist.

(a) Note that $a>1$ ensures $a^n < a^{n+1}$, so the sequence is strictly increasing.

Thus, it suffices to prove a^n is unbounded above. Assume, for the sake of contradiction that a^n is bounded above. Then it must converge to some $L \in \mathbb{R}$. Thus

$$L = \lim_{n\to\infty} a^n = \underbrace{\lim_{n\to\infty} a^{n+1}}_{\text{subsequence of convergent sequence has same limit}} = \lim_{n\to\infty} a^n a = La.$$

$L=0$ is impossible since a^n is strictly increasing and $a>1$. Thus $L=La$ implies $a=1$, which is a contradiction. This shows a^n is unbounded above.

(b) First, we show $\lim_{n\to\infty} a^n \neq \pm\infty$.

Since the odd elements are negative and the even elements are positive, for $M=0$, there does not exist N s.t. $n \geq N$ ensures either $a^n \geq M$ or $a^n \leq M$. Thus $\lim_{n\to\infty} a^n \neq \pm\infty$.

Now, we show a^n does not converge.
By the previous part, $\lim_{n\to\infty} \underbrace{|a^n|}_{=|a|^n} = +\infty$.
Thus a^n is not a bounded sequence.
Hence, it cannot converge.

Question 2:

ⓐ Fix $\varepsilon > 0$. Note that, by the reverse triangle inequality,

$$\big||t| - |t_n|\big| \leq |t - t_n|.$$

Since $\lim_{n\to\infty} t_n = t$, $\exists\ N \in \mathbb{R}$ s.t. $n > N$ ensures $|t - t_n| < \varepsilon$, hence $\big||t| - |t_n|\big| < \varepsilon$.
Since $\varepsilon > 0$ was arbitrary, this shows $\lim |t_n| = |t|$.

ⓑ The converse is not true.
Let $t_n = (-1)^n$. Then $|t_n| = 1$ is a convergent sequence, but t_n is not a convergent sequence.

Question 3

Since $\lim_{n\to\infty} s_n = s \neq 0$, $\exists\, N$ s.t. $n > N$ ensures $|s_n - s| < \frac{|s|}{2}$. By the reverse triangle inequality, this shows that $n > N$ ensures $|s| - |s_n| < \frac{|s|}{2} \iff \underbrace{\frac{|s|}{2} < |s_n|}_{(*)}$.

First, we will show $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$. Note that $\frac{1}{s_n}$ is well-defined for all $n > N$ by $(*)$. Fix $\varepsilon > 0$. Note that, for $n > N$,

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s - s_n}{s_n s}\right| = \frac{|s - s_n|}{|s_n||s|} \overset{(*)}{<} \frac{|s - s_n| 2}{|s|^2}.$$

Since $s_n \to s$, $\exists\, \tilde{N}$ s.t. $n > \tilde{N}$ ensures $|s - s_n| < \frac{\varepsilon |s|^2}{2}$. Thus $n > \max\{N, \tilde{N}\}$ ensures $\left|\frac{1}{s_n} - \frac{1}{s}\right| < \varepsilon$. This shows $\frac{1}{s_n} \to \frac{1}{s}$.

The result of Q3 then follows from the fact that the limit of the product of convergent sequences is the product of the limits.

Question 4

(10) If $\lim_{n\to\infty} r_n = -\infty$, the result is immediate.
Thus, it remains to consider the remaining cases.

Case 1: Suppose $\lim_{n\to\infty} r_n = r \in \mathbb{R}$

Case 1a: If $\lim_{n\to\infty} t_n = +\infty$, we are done.

Case 1b: Suppose $\lim_{n\to\infty} t_n = t \in \mathbb{R}$. Assume for the sake of contradiction that $t<r$. Let $\varepsilon = \frac{r-t}{2} > 0$. Then $\exists\, N_r, N_t$ s.t. $n > N_r$ ensures $|r_n - r| < \varepsilon$ and $n > N_t$ ensures $|t_n - t| < \varepsilon$. Let $N = \max\{N_r, N_t\}$. Then $n > N$ ensures

$$t_n < t + \varepsilon = t + \frac{r-t}{2} = \frac{t+r}{2} = r - \frac{r-t}{2} = r - \varepsilon < r_n.$$

This contradicts that $r_n \le t_n \ \forall n \in \mathbb{N}$.
Thus $\lim_{n\to\infty} t_n = t \geq r = \lim_{n\to\infty} r_n$.

Case 1c: Suppose $\lim_{n\to\infty} t_n = -\infty$. Then $\exists\, N_t$ s.t. $\forall n > N_t$, $t_n < r-1$. There also exists N_r s.t. $\forall n > N_t$, $r-1 < r_n$. Thus for $N = \max\{N_t, N_r\}$, $n > N$ ensures

$$t_n < r-1 < r_n.$$

Again, this contradicts that $r_n \le t_n \ \forall n \in \mathbb{N}$.
Thus $\lim_{n\to\infty} t_n = -\infty$ is impossible.

Case 2: Suppose $\lim_{n\to\infty} r_n = +\infty$. Fix $M > 0$. Then $\exists$ N s.t. $\forall n \geq N$, $M < r_n \le t_n$. This shows $\lim_{n\to\infty} t_n = +\infty$.

Question 5

(a) Let $s := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

If $s=+\infty$, then $\forall M>0$, $\exists N$ s.t. $n>N$ ensures $M < a_n \le s_n$, so $\lim_{n\to\infty} s_n = +\infty$.

If $s=-\infty$, then $\forall M<0$, $\exists N$ s.t. $n>N$ ensures $s_n \le b_n < M$, so $\lim_{n\to\infty} s_n = -\infty$.

If $s\in\mathbb{R}$, then $\forall \varepsilon>0$, $\exists N_a, N_b$ s.t. $n>N_a$ ensures $|a_n - s| < \varepsilon$ and $n>N_b$ ensures $|b_n - s|<\varepsilon$. Thus, $n>\max\{N_a, N_b\}$ ensures $s-\varepsilon < a_n \le s_n \le b_n < s+\varepsilon$, so $\lim_{n\to\infty} s_n = s$.

(b) Since $-t_n \le s_n \le t_n$ and $\lim_{n\to\infty} t_n = 0$ ensures $\lim_{n\to\infty} -t_n = 0$, the result is a consequence of the Squeeze Lemma.

(c) No. Consider the sequences $s_n = (0,0,0,\ldots)$ and $t_n = (1,1,1,\ldots)$.

Question 6

Case 1: $\lim_{n\to+\infty} s_n \in \mathbb{R}$
Since $t_n = (k, k, k, \ldots)$ is a sequence that converges to k and s_n is a convergent sequence, by the theorem that the limit of the product is the product of the limits,

$$\lim_{n\to+\infty} ks_n = \lim_{n\to+\infty} t_n s_n = \left(\lim_{n\to+\infty} t_n\right)\left(\lim_{n\to+\infty} s_n\right) = k \lim_{n\to+\infty} s_n.$$

Case 2:$\lim_{n\to+\infty} s_n = \pm\infty$ and $k = 0$
Then $ks_n = (0, 0, 0, \ldots)$ converges to $0 = k \cdot (+\infty) = k \lim_{n\to+\infty} s_n$.

Case 3a: $\lim_{n\to+\infty=+\infty}$ and $k > 0$
We must show that ks_n diverges to $+\infty$. Fix $M > 0$. Since s_n diverges to ∞, there exists N so that $n >$ ensures $s_n > M/k \implies ks_n > M$. This shows $\lim_{n\to+\infty} ks_n = +\infty$.

Case 3b: $\lim_{n\to+\infty=+\infty}$ and $k < 0$
Then $-(ks_n) = (-k)s_n$. By Case 3a, $\lim_{n\to+\infty}(-k)s_n = +\infty$. By Q12(b), this implies $\lim_{n\to+\infty} ks_n = -\infty$.

Case 4a: $\lim_{n\to+\infty} s_n = -\infty$ and $k > 0$
Then $-(ks_n) = k(-s_n)$. By Q12(b), $\lim_{n\to+\infty} -s_n = +\infty$. Thus, Case 3a ensures $\lim_{n\to+\infty} k(-s_n) = +\infty$. Thus, by Q12 again, $\lim_{n\to+\infty} ks_n = -\infty$.

Case 4b: $\lim_{n\to+\infty} s_n = -\infty$ and $k < 0$
Then $-(ks_n) = (-k)s_n$. By Case 4a, $\lim_{n\to+\infty}(-k)s_n = -\infty$. By Q12(b), this implies $\lim_{n\to+\infty} ks_n = +\infty$.

Question 7

We construct the sequence inductively. First, by density of $\mathbb{Q}$ in $\mathbb{R}$, $\exists$ $r_1 \in \mathbb{Q}$ s.t. $a-1 < r_1 < a$.

Suppose we have defined the first n terms of the sequence to be increasing, rational, and satisfy $a - \frac{1}{k} < r_k < a$ for $k = 1, \dots, n$.

By density of $\mathbb{Q}$ in $\mathbb{R}$, $\exists$ $r_{n+1} \in \mathbb{Q}$ s.t. $\max\{a - \frac{1}{n+1}, r_n\} < r_{n+1} < a$.

In this way, we obtain an increasing sequence r_n of rational numbers satisfying
$$a - \frac{1}{n} < r_n < a.$$
By the Squeeze Lemma, $\lim_{n\to\infty} r_n = a$.

Question 8

The converse is...

"Consider a sequence a_n and suppose $\lim_{n\to\infty} a_n = a$. Then $\lim_{n\to\infty} a_{2n} = a = \lim_{n\to\infty} a_{2n-1}$."

Fix $\varepsilon > 0$. Since $a_n \to a$, $\exists N$ s.t. $n > N$ ensures $|a_n - a| < \varepsilon$. Since $2n > n$ and $2n-1 \geq n$ $\forall n \in \mathbb{N}$, $n > N$ also ensures $|a_{2n} - a| < \varepsilon$ and $|a_{2n-1} - a| < \varepsilon$. Thus

$$\lim_{n\to\infty} a_{2n} = a = \lim_{n\to\infty} a_{2n-1}.$$

Question 9

First suppose $r \geq 1$. Then, $\forall n \in \mathbb{N}$, $r^{n+1} = r^n r \geq r^n \cdot 1 = r$. Likewise, since $(\sqrt[n]{r})^n = \underbrace{(\sqrt[n]{r}) \cdot \ldots \cdot (\sqrt[n]{r})}_{n \text{ times}} = 1$,

we must have $\sqrt[n]{r} \geq 1$. Likewise,

$$\left(\sqrt[n+1]{r}\right)^{n+1} = r \leq \left(\sqrt[n]{r}\right)^n \sqrt[n]{r} = \left(\sqrt[n]{r}\right)^{n+1},$$

so $\sqrt[n+1]{r} \leq \sqrt[n]{r}$.

This shows that, when $a \geq 1$, $a^{1/n}$ is a decreasing sequence bounded below by 1; thus, it must converge to $L \geq 1$. Since the limit of the product is the product of the limits, $a^{2/n}$ must converge to $L^2 \geq 1$. By Q7, the even elements of the sequence $a^{2/n}$ must converge to the same limit.

However, the even elements of the sequence $a^{1/n}$ are exactly the elements of the sequence $(a^{1/n})$. This shows $L^2 = L \geq 1$, so $L = 1$.

Now, suppose $a \in (0,1)$. Then $\frac{1}{a} \geq 1$, so $a^{-1/n} \to 1$. Since the limit of the quotient of convergent sequences (with nonzero denominator) is the quotient of the limit, $a^{1/n} \to 1$.