Math 117: Homework 5

Due Monday, May 12th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1*

Find the limit of the following sequences. Justify your answer.

(i) $\lim_{n\to+\infty} \frac{2n^2+n+3}{n^2+1}$

(ii) $\lim_{n\to+\infty} \sqrt{n+1} - \sqrt{n}$

(iii) $\lim_{n\to+\infty} \frac{1}{n!}$

Question 2*

Define a sequence s_n as follows: $s_1 = 1$ and, for $n \geq 1$, $s_{n+1} = \frac{1}{3}(s_n + 1)$.

(a) Use induction to prove that $s_n \geq 1/2$ for all n.

(b) Use the definition of the sequence and part (a) to show that the sequence is decreasing.

(c) Prove that $\lim s_n = s$ for some $s \in \mathbb{R}$.

(d) Prove that $\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} s_{n+1}$.

(e) Use part (d), the definition of s_n, and the limit theorems to find the value of s.

Question 3*

Let s_n be an increasing sequence of positive numbers and define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$. Prove σ_n is an increasing sequence.

Question 4*

Prove $\limsup_{n\to+\infty} |s_n| = 0$ if and only if $\lim_{n\to+\infty} s_n = 0$.

Question 5

Given a sequence s_n, prove that $\limsup_{n\to+\infty} |s_n| < +\infty$ if and only if s_n is bounded.

Question 6

Determine whether the following statements are true or false. If they are true, prove them. If they are false, give a counterexample and justify it.

(a) If a sequence s_n satisfies $\limsup_{n\to+\infty} s_n = 2$, then $s_n > 1.99$ for all n large enough.

(b) If a sequence s_n satisfies $\limsup_{n\to+\infty} s_n = b$, then $s_n \leq b$ for all n large enough.

Question 7*

Give examples of...

(a) A sequence x_n of irrational numbers having a limit that is a rational number.

(b) A sequence r_n of rational numbers having a limit that is an irrational number.

Justify your answers.

Question 8*

Let s_n and t_n be the following sequences that repeat in cycles of four:

$$s_n = (0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, \dots)$$
$$t_n = (2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, 2, \dots).$$

Compute the following quantities. You do not need to justify your answer.

(a) $\liminf s_n$

(b) $\limsup s_n$

(c) $\liminf t_n$

(d) $\limsup t_n$

(e) $\liminf s_n + \liminf t_n$

(f) $\liminf s_n + \limsup t_n$

(g) $\limsup s_n + \limsup t_n$

(h) $\limsup(s_n t_n)$

(i) $\liminf(s_n + t_n)$

(j) $\limsup(s_n + t_n)$

(k) $\liminf(s_n t_n)$

Pay attention to the above examples where we see that the liminf and limsup of the sum/product is not necessarily equal to the sum/product of the liminf and limsup.

Question 9

Determine if the following statement is true or false. If it is true, prove it. If it is false, provide a counterexample.

If s_n and t_n are sequences whose limits exist and for which $s_n < t_n$ for all but finitely many $n \in \mathbb{N}$, then $\lim_{n\to+\infty} s_n < \lim_{n\to+\infty} t_n$.

Question 10

Determine if the following statement is true or false. If it is true, prove it. If it is false, provide a counterexample.

If a sequence a_n is not increasing, then a_n is decreasing.

Question 11*

(i) Prove the nested interval theorem: If $\{[a_n, b_n]\}_{n\in\mathbb{N}}$ is a collection of closed intervals satisfying

$$[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n], \text{ for all } n \in \mathbb{N},$$

then $\cap_{n\in\mathbb{N}}[a_n, b_n]$ is nonempty.

(ii) Find a necessary and sufficient condition that$\cap_{n\in\mathbb{N}}[a_n, b_n]$ contains a single point. Justify your answer with a proof.