Homework 5 Solutions

(1)

(i) $\lim_{n\to\infty} \frac{2n^2+n+3}{n^2+1} = \lim_{n\to\infty} \frac{2+\frac{1}{n}+\frac{3}{n^2}}{1+\frac{1}{n^2}}$

$= 2,$

since the limit theorems (sum, product) ensure the numerator and denominator each converge and the limit of the denominator is nonzero.

(ii) $\lim_{n\to\infty} \sqrt{n+1}-\sqrt{n} = \lim_{n\to\infty} \frac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}}$

$= \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0,$

since the denominator is a positive sequence that diverges to $+\infty$.

(iii) Since $0 \leq \frac{1}{n!} \leq \frac{1}{n}$, by the Squeeze lemma, $\lim_{n\to\infty} \frac{1}{n!} = 0$.

(2)

(a) Base case: When $n=1$, $s_1 = 1 \geq \frac{1}{2}$.

Inductive step: Suppose $s_n \geq \frac{1}{2}$. We aim to show $s_{n+1} \geq \frac{1}{2}$. By definition $s_{n+1} = \frac{1}{3}(s_n + 1)$.

Since $s_n \geq \frac{1}{2}$, $s_n + 1 \geq \frac{3}{2}$, so

$$s_{n+1} = \frac{1}{3}(s_n + 1) \geq \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}.$$

This completes the proof.

(b) We aim to show $s_{n+1} \leq s_n$ for all $n \in \mathbb{N}$.

By part (a), $s_n \geq \frac{1}{2}$, so $\frac{2}{3} s_n \geq \frac{1}{3}$. Thus, by definition of the sequence,

$$s_{n+1} = \frac{1}{3}(s_n + 1) = \frac{s_n}{3} + \frac{1}{3} \overset{(a)}{\leq} \frac{s_n}{3} + \frac{2 s_n}{3} = s_n,$$

which completes the proof.

(c) Since s_n is a decreasing sequence, $s_1 \geq s_n$ $\forall n \in \mathbb{N}$. Since $s_n \geq \frac{1}{2}$ $\forall n \in \mathbb{N}$, we have $\frac{1}{2} \leq s_n \leq s_1 = 1$ $\forall n \in \mathbb{N}$. Thus s_n is a bounded, decreasing sequence. Since all bounded monotone sequences converge, $\lim_{n \to \infty} s_n = s$ for some $s \in \mathbb{R}$.

(d) We will show $\lim_{n\to\infty} s_{n+1} = s$. Fix $\varepsilon > 0$.

By (c), $\exists N$ s.t. $n > N$ ensures $|s_n - s| < \varepsilon$. Thus $|s_{n-1} - s| < \varepsilon$.
This shows $\lim_{n\to\infty} s_{n+1} = s$.

(e) By part (d) and the limit theorems,

$$s = \lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} \frac{1}{3}(s_n + 1) = \frac{1}{3}(s+1).$$

Thus, $\frac{2}{3}s = \frac{1}{3}$, so $s = \frac{1}{2}$.

(3) We must show
$\sigma_{n+1} = \frac{1}{n+1}(s_1 + s_2 + \dots + s_{n+1}) \geq \frac{1}{n}(s_1 + s_2 + \dots + s_n) = \sigma_n$,
which is equivalent to showing
$(s_1 + s_2 + \dots + s_{n+1}) \geq \frac{n+1}{n}(s_1 + s_2 + \dots + s_n) = (1 + \frac{1}{n})(s_1 + \dots + s_n)$.
$= s_1 + \dots + s_n + \frac{1}{n}(s_1 + \dots + s_n)$

Subtracting $s_1 + \dots + s_n$ from both sides shows this is equivalent to showing $s_{n+1} \geq \frac{1}{n}(s_1 + \dots + s_n)$. Multiplying both sides by n, this is equivalent to $n s_{n+1} \geq s_1 + \dots + s_n$.
Since s_n is increasing, $s_{n+1} \geq s_i \;\forall i = 1, \dots, n$ which gives the result.

④ First, suppose $\lim s_n = 0$. By HW4, Q2, $\lim |s_n| = 0$, so $\limsup |s_n| = \liminf |s_n| = \lim |s_n| = 0$.

Now suppose $\limsup |s_n| = 0$. By definition, this implies $\lim_{N\to\infty} a_N = 0$, where $a_N = \sup\{|s_n| : n > N\}$. Fix $\varepsilon > 0$, and choose N_0 so that $N > N_0$ ensures $|a_N - 0| < \varepsilon \Leftrightarrow |a_N| < \varepsilon$. $\Leftrightarrow a_N < \varepsilon$, since a_N is nonnegative. In particular, $a_{N_0+1} < \varepsilon$, so by definition of a_{N_0}, we have that $n > N_0 + 1$ ensures $|s_n| < \varepsilon$. Therefore $\lim_{n\to\infty} s_n = 0$.

⑤ Assume s_n is a bounded sequence. Then $\exists\, M_0$ s.t. $|s_n| \le M_0$ $\forall n \in \mathbb{N}$. Hence $\sup\{|s_n| : n > N\} \le M_0$ $\forall N \in \mathbb{N}$. Thus, $\lim_{N\to\infty} \sup\{|s_n| : n > N\} \le M_0$, so $\limsup_{n\to\infty} |s_n| \le M_0 < +\infty$.

Now, assume $\limsup_{n\to\infty} |s_n| < +\infty$. Recall that

$$\limsup_{n\to\infty} |s_n| = \lim_{N\to\infty} \underbrace{\sup\{|s_n| : n > N\}}_{a_N}.$$

Since a_N is a convergent sequence, it is bounded, and $\exists\, M_0$ s.t. $|a_N| \le M_0$ $\forall N \in \mathbb{N}$. In particular, $|a_1| \le M_0 \Leftrightarrow |\sup\{|s_n| : n > 1\}| \le M_0$

so $|s_n| \le \max\{|s_1|, M_0\}$. Thus s_n is a bounded sequence.

⑥ ⓐ False. Consider: $s_n = (-1)^n 2$.

Then $\limsup_{n\to\infty} s_n = \lim_{N\to\infty} \sup\{s_n : n > N\}$

$= \lim_{N\to\infty} 2 = 2$.

However, all odd elements of s_n are strictly less than 1.99.

ⓑ False. Consider $s_n = b + \frac{1}{n}$.

Since s_n is convergent,

$\lim_{n\to\infty} s_n = b = \limsup_{n\to\infty} s_n$.

However $s_n > b$ for all n.

⑦ ⓐ Define $x_n = \frac{\sqrt{2}}{n}$. As shown in class, $\sqrt{2}$ is an irrational number. Since $\mathbb{Q}$ is a field, the product of two rational numbers is a rational number. Since $\mathbb{N} \subseteq \mathbb{Q}$ and $x_n \cdot n = \sqrt{2} \notin \mathbb{Q}$, we must have that $x_n \notin \mathbb{Q}$, so x_n is a sequence of irrational numbers.

Claim: $\lim_{n\to\infty} x_n = 0$. We must show that

for all $\varepsilon>0$, there exists N s.t. $n>N$ ensures $|x_n|<\varepsilon$. Note that $|x_n| = \left|\frac{\sqrt{2}}{n}\right| = \frac{\sqrt{2}}{n} < \varepsilon \iff \frac{\sqrt{2}}{\varepsilon} < n$.
Therefore, for all $\varepsilon>0$, if we take $N=\frac{\sqrt{2}}{\varepsilon}$, then for all $n>N$, $|x_n|<\varepsilon$.

(b) Define $r_n = 1.\underbrace{41421\ldots}_{\text{first } n \text{ digits of decimal approximation of } \sqrt{2}}$.

Or more precisely, we define r_n by $r_n = \lfloor \sqrt{2}\cdot 10^n \rfloor / 10^n$, where $\lfloor a \rfloor$ represents the largest integer less than or equal to a. Then $r_n \in \mathbb{Q}$.

Claim: $\lim_{n\to\infty} r_n = \sqrt{2}$. Note that $|r_n - \sqrt{2}| = 10^{-n} \left| \lfloor \sqrt{2}\cdot 10^n \rfloor - \sqrt{2}\cdot 10^n \right| \le 10^{-n}$, and $10^{-n} < \varepsilon \iff \frac{1}{\varepsilon} < 10^n \iff \log_{10}\frac{1}{\varepsilon} < n$.
Therefore, for all $\varepsilon>0$, if we take $N = \log_{10}\frac{1}{\varepsilon}$, then for all $n>N$, $|r_n - \sqrt{2}| < \varepsilon$.

8)

a) 0

b) 2

c) 0

d) 2

e) 0+0 = 0

f) 0+2 = 2

g) 2+2 = 0

h) 2

i) 1

j) 3

k) 0

(9) False. Let $s_n = (0, 0, 0, \dots)$ and $t_n = \frac{1}{n}$. Then $s_n < t_n$ for all $n \in \mathbb{N}$, hence all but finitely many n. However $\lim s_n = \lim t_n$.

(10) False. Consider $a_n = (-1)^n$. This is neither increasing nor decreasing.

(11) Since $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$, we have that a_n is an increasing sequence bounded above by $b_m, m \in \mathbb{N}$, and b_n is a decreasing sequence bounded below by $a_m, m \in \mathbb{N}$. Thus both converge, and we denote

their limits by a and b.

Since $a_n \le b_m$ for all $n, m \in \mathbb{N}$, $a_n \le b$ and $a \le b_m$ $\forall\, n, m \in \mathbb{N}$. Thus $a \le b$ and

$$[a,b] \subseteq [a_n, b_n] \quad \forall\, n \in \mathbb{N}.$$

This shows $[a,b] \subseteq \bigcap_{n=1}^{\infty} [a_n, b_n]$, so the set on the RHS is nonempty.