Math 117: Homework 6

Due Monday, May 19th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1*

Consider the sequences defined as follows:

$$a_n = (-1)^{n+1}, \quad b_n = -\frac{1}{n}, \quad c_n = 2n.$$

(a) For each sequence, give an example of a monotone subsequence.

(b) For each sequence, give its set of subsequential limits. Justify your answer.

(c) For each sequence, give its lim inf and lim sup. Justify your answer.

(d) Which of the sequences converges? Diverges to $+\infty$? Diverges to $-\infty$? (You do not need to justify your answer.)

(e) Which of the sequences is bounded? (You do not need to justify your answer.)

Question 2

(a) State the definition of convergence for a sequence s_n to a limit s.

(b) State what it means for a sequence s_n to *not* converge to a limit s by negating the definition from part (a).

(c) Suppose that s_n does *not* converge to $s \in \mathbb{R}$. Prove that there exists $\epsilon > 0$ and a subsequence s_{n_k} so that $|s_{n_k} - s| \geq \epsilon$ for all k.

Question 3*

One can show that the set of rational numbers $\mathbb{Q}$ can be listed as a sequence r_n. The exact procedure is a little tedious, but you can get an idea of how it works by considering the below diagram from the textbook. For example, $r_1 = 0, r_2 = 1, r_3 = 1/2$, and so on. Note that some numbers, such as -1, are included multiples times.

(a) For any $\epsilon > 0$ and $a \in \mathbb{R}$, show that the set $\{r \in \mathbb{Q} : |r - a| < \epsilon\}$ contains infinitely many elements.

(b) Let r_n be the sequence of rational numbers. Use part (a) to show that for any $a \in \mathbb{R}$, there exists a subsequence r_{n_k} that converges to a.

(c) Let r_n be the sequence of rational numbers. Show that there exists a subsequence r_{n_k} satisfying $\lim_{k\to+\infty} r_{n_k} = +\infty$.

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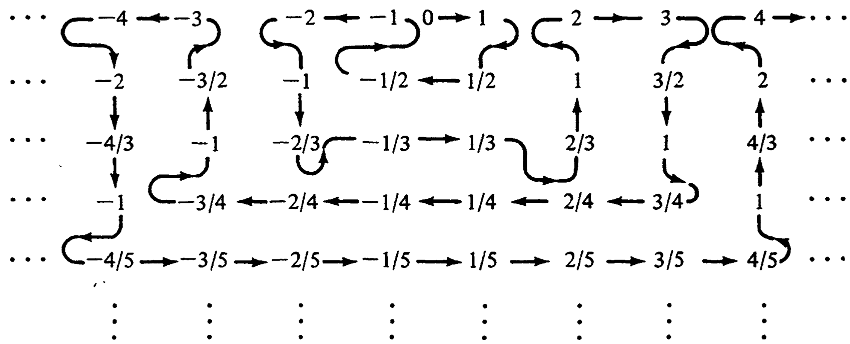

Background on Infinite Series

In calculus, you encountered infinite series of the form

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \dots .$$

In fact, these are just limits of sequences. In particular, if we define the sequence

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \dots + a_n$$

to be the sum of the first n terms of the series, then we define the value of the infinite series to be

$$\sum_{k=1}^{\infty} a_k = \lim_{n \to +\infty} s_n .$$

DEFINITION 1. Given a series $\sum_{k=1}^{\infty} a_k$, define the sequence $s_n = \sum_{k=1}^{n} a_k$. Then the series $\sum_{k=1}^{\infty} a_k$ *converges* to a number L if and only if the sequence s_n converges to L. Likewise, the series *diverges to* $+\infty$ *or* $-\infty$ if and only if the sequence s_n diverges to $+\infty$ or $-\infty$.

Question 4* (Cauchy criterion)

Recall that a sequence s_n is a *Cauchy sequence* if

$$\text{for all } \epsilon > 0, \text{ there exists } N \in \mathbb{R} \text{ so that } n, m > N \text{ ensures } |s_n - s_m| < \epsilon.$$

(a) Prove that the following is an equivalent definition of a Cauchy sequence:

s_n is a Cauchy sequence if, for all $\epsilon > 0$, there exists $N \in \mathbb{R}$ so that $n > m > N$ ensures $|s_n - s_m| < \epsilon$.

(b) Prove the following theorem about series, known as the Cauchy criterion.

THEOREM 1 (Cauchy Criterion). *A series $\sum_{k=1}^{\infty} a_k$ is convergent if and only if*

$$\textit{for all } \epsilon > 0 \textit{ there exists } N \in \mathbb{R} \textit{ so that } n > m > N \textit{ ensures } \left| \sum_{k=m+1}^{n} a_k \right| < \epsilon.$$

(c) Now use Theorem 1 to prove the following corollary:

COROLLARY 2. *If a series $\sum_{k=1}^{\infty} a_k$ is convergent, then $\lim_{k \to +\infty} a_k = 0$.*

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Question 5

(a) Prove the following by induction: for $a \neq 1$,

$$\sum_{i=0}^{m-1} a^i = 1 + a + a^2 + \cdots + a^{m-1} = \frac{1 - a^m}{1 - a}.$$

(b) Use part (a) to show that

$$\sum_{i=n}^{m-1} a^i = a^n + a^{n+1} + \cdots + a^{m-2} + a^{m-1} = \frac{a^n - a^m}{1 - a}.$$

(c) Recall that, by the triangle inequality,

$$\left| \sum_{i=1}^{n} a_i \right| = |a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n| = \sum_{i=1}^{n} |a_i|.$$

Let s_n be a sequence such that $|s_{n+1} - s_n| \leq 4^{-n}$ for all $n \in \mathbb{N}$. Use part (b) and the above inequality to prove s_n is a Cauchy sequence.

(d) Does the sequence from part (c) converge? Justify your answer.

Question 6* (decimal expansions)

In this problem you will show that any number that can be represented as a nonnegative decimal expansion can be thought of as the limit of a bounded increasing sequence of real numbers. Since all bounded monotone sequences converge, this guarantees that any decimal expansion you can imagine represents (converges to) a real number.

Suppose we are given a decimal expansion $K.d_1d_2d_3d_4\ldots$, where K is a nonnegative integer and each $d_j \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let

$$s_n = K + \frac{d_1}{10^1} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}.$$

(a) Show s_n is an increasing sequence. (This is almost obvious. Your proof should be short.)

(b) Use the result from Q5(a) to prove that $\frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} = 1 - \frac{1}{10^n}$.

(c) Use part (b) to prove that s_n is a bounded sequence.

(d) Since $0.\bar{9} = 0.999\ldots$ and 1 are both decimal expansions, by what you have shown, they both correspond to a real number. Use the hint from part (b) to show that they actually correspond to the same real number.

Question 7 (geometric series)

(a) Prove that for $|r| < 1$, $\sum_{k=1}^{\infty} r^k = \frac{1}{1-r}$.

(b) Prove that for $|r| \geq 1$, $\sum_{k=1}^{\infty} r^k$ does not converge. (**Hint**: Use Corollary 2 from Q4.)

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Question 8*

Let s_n be a sequence of nonnegative numbers, and for each n define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$.

(a) Show $\liminf s_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup s_n$.
(**Hint:** For the first inequality, show that $M > N$ implies

$$\inf\{\sigma_n : n > M\} \geq \left(1 - \frac{N}{M}\right) \inf\{s_n : n > N\}.$$

For the last inequality, show first that $M > N$ implies

$$\sup\{\sigma_n : n > M\} \leq \frac{1}{M}(s_1 + s_2 + \cdots + s_N) + \sup\{s_n : n > N\}.)$$

(b) Show that if $\lim s_n$ exists, then $\lim \sigma_n$ exists and $\lim \sigma_n = \lim s_n$.

(c) Give an example for which $\lim \sigma_n$ exists but $\lim s_n$ does not exist.

Question 9

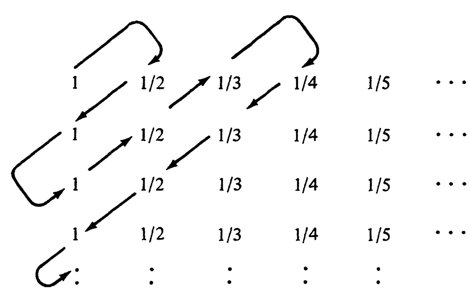
Suppose s_n and t_n are bounded sequences.

(a) Prove that $\limsup s_n + t_n \leq \limsup s_n + \limsup t_n$.

(b) Give an examples of bounded sequences s_n and t_n for which $\limsup s_n + t_n < \limsup s_n + \limsup t_n$.

Question 10*

Let s_n be the sequence defined in the following figure from the textbook:



(a) Find the set S of subsequential limits of s_n. Justify your answer.

(b) Determine $\limsup s_n$ and $\liminf s_n$. Justify your answer.

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Question 11

(a) Suppose s_n has a subsequence s_{n_k} that is bounded. Show that this implies s_n has a convergent subsequence.

(b) Suppose that s_n has no convergent subsequences. Prove that $\lim_{n\to+\infty} |s_n| = +\infty$.

(Hint: prove the result by contradiction, by showing that if $\lim_{n\to+\infty} |s_n| \neq +\infty$, then s_n has a bounded subsequence.)

Question 12*

In this problem, we will consider sequences s_n satisfying the following property:

$$\exists s \in \mathbb{R} \text{ s.t. every subsequence } s_{n_k} \text{ of } s_n \text{ has a further subsequence } s_{n_{k_l}} \text{ satisfying } \lim_{l\to+\infty} s_{n_{k_l}} = s. \quad (*)$$

(a) Prove that if $\lim s_n = s$, then property (*) holds.

(b) Prove that if property (*) holds, then $\lim s_n = s$. (Hint: Use HW6, Q2, part c)

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