Homework 6 Solutions

(1)

(a)

Sequence	Monotone subsequence
a_n	$(1, 1, 1, \ldots)$
b_n	$-\frac{1}{n}$
c_n	$2n$

(b)

Sequence	Set	Justification
a_n	$\{-1, 1\}$	1 and -1 are clearly subsequential limits, since the constant sequences $(1, 1, 1, \ldots)$ and $(-1, -1, -1, \ldots)$ are subsequences of a_n.

Fix $t \in \mathbb{R} \setminus \{-1, 1\}$. Let $\varepsilon = \min\{|t-1|, |t-(-1)|\}$. Then $\varepsilon > 0$, and $\{n : |(-1)^n - t| < \varepsilon\} = \emptyset$. By the main subsequences theorem, this implies that t is not a subsequential limit.

Sequence	Set	Justification
b_n	$\{0\}$	If a sequence has a limit, then all subsequences have the same limit
c_n	$\{+\infty\}$	

(c) $\limsup_{n\to\infty} a_n = \lim_{N\to\infty} \sup\{a_n : n > N\} = \lim_{N\to\infty} 1 = 1$

$\liminf_{n\to\infty} a_n = \lim_{N\to\infty} \inf\{a_n : n > N\} = \lim_{N\to\infty} -1 = -1$

Since the limits of b_n, c_n exist, their limsup's and liminf's must coincide with their limits. Thus,

$\limsup_{n\to\infty} b_n = \liminf_{n\to\infty} b_n = 0$

$\limsup_{n\to\infty} c_n = \liminf_{n\to\infty} c_n = +\infty$

(d) a_n does not converge, since its set of subsequential limits contains more than one element. It also does not diverge to $+/-\infty$, since it is bounded.

b_n converges to 0

c_n diverges to $+\infty$

(e) $|a_n| \le 1 \ \forall n \in \mathbb{N}$, so it is bounded.
b_n and d_n are convergent, hence bounded.

c_n is not bounded, since it diverges to $+\infty$.

(2) (a) A sequence s_n converges to a limit s if for all $\varepsilon > 0$, $\exists N$ s.t. $n > N$ ensures $|s_n - s| < \varepsilon$.

(b) A sequence s_n doesn't converge to a limit s if $\exists\, \varepsilon > 0$ s.t. $\forall N$, $\exists n > N$ s.t. $|s_n - s| \geq \varepsilon$

(c) We construct such a subsequence.
Taking $N = 1$ in part (b), $\exists\, n_1 > 1$ s.t. $|s_{n_1} - s| \geq \varepsilon$. Suppose we have chosen n_{k-1}. Taking $N = n_{k-1}$ in part (b), $\exists\, n_k > n_{k-1}$ s.t. $|s_{n_k} - s| \geq \varepsilon$.

Therefore there exists a subsequence s_{n_k} s.t. $|s_{n_k} - s| \geq \varepsilon$ $\forall k$.

③ ⓐ We must show that for all $\varepsilon>0$ and $a\in\mathbb{R}$, $S=\{r\in\mathbb{Q}: a-\varepsilon<r<a+\varepsilon\}$ is infinite. We proceed by induction. By denseness of $\mathbb{Q}$ in $\mathbb{R}$, there exists $r_1\in\mathbb{Q}$ so that $a-\varepsilon<r_1<a+\varepsilon$, so $r_1\in S$. By denseness of $\mathbb{Q}$ in $\mathbb{R}$, there exists $r_2\in\mathbb{Q}$ so that $a-\varepsilon<r_2<r_1<a+\varepsilon$, so $r_2\in S$.
Assume we have picked k distinct elements $r_1,r_2,\ldots,r_k\in S$ satisfying $r_k<r_{k-1}<\ldots<r_2<r_1$.

By denseness of $\mathbb{Q}$ in $\mathbb{R}$, there exists $r_{k+1}\in\mathbb{Q}$ so that $a-\varepsilon<r_{k+1}<r_k<\ldots<r_1<a+\varepsilon$, so $r_{k+1}\in S$. Thus S has infinitely many elements.

ⓑ Since $\{r\in\mathbb{Q}: |r-a|<\varepsilon\}$ contains infinitely many elements and r_n is the sequence of rational numbers, $\{n\in\mathbb{N}: |r_n-a|<\varepsilon\}$ is infinite for all $\varepsilon>0$.
By the main subsequences theorem, this ensures that there is a subsequence r_{n_k} that converges to a.

(c) Since r_n is unbounded above, the main subsequences theorem ensures that there is a subsequence that diverges to $+\infty$.

(4)

(a) Suppose s_n is a Cauchy sequence, according to our definition from class. Fix $\varepsilon>0$. Then there exists N s.t. $n,m>N$ ensures $|s_n-s_m|<\varepsilon$. In particular, if $n>m>N$, we have $|s_n-s_m|<\varepsilon$.

Now, suppose s_n is a Cauchy sequence, according to the new definition. Fix $\varepsilon>0$. Then $\exists N$ s.t. $k>l>N$ ensures $|s_k-s_l|<\varepsilon$. Suppose $n,m>N$. If $n=m$, then $|s_n-s_m|=0<\varepsilon$. If $n>m$, take $k=n$, $l=m$ to see $|s_n-s_m|<\varepsilon$. Lastly, if $n<m$, take $k=m$, $l=n$ to see $|s_n-s_m|<\varepsilon$.

④ⓑ

$\sum_{k=1}^{\infty} a_k$ is convergent

$\Updownarrow$

$s_n = \sum_{k=1}^{n} a_k$ converges

$\Updownarrow$

s_n is Cauchy

$\Updownarrow$ ⓐ

$\forall \varepsilon > 0$, $\exists N \in \mathbb{R}$ so that $n > m > N$ ensures $|s_n - s_m| < \varepsilon$

$$\sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k = \sum_{k=m+1}^{n} a_k$$

$\Updownarrow$

$\forall \varepsilon > 0$, $\exists N \in \mathbb{R}$ so that $n > m > N$ ensures $\left|\sum_{k=m+1}^{n} a_k\right| < \varepsilon$

ⓒ Suppose $\sum_{k=1}^{\infty} a_k$ is convergent. WTS $\lim a_k = 0$. Fix $\varepsilon > 0$. By part ⓑ, $\exists N$ s.t. $n > m > N$ implies $\left|\sum_{k=m+1}^{n} a_k\right| < \varepsilon$. In particular, $\exists N$ s.t. $m > N$ and $n = m+1$ implies $|a_n| < \varepsilon$, so $|a_n - 0| < \varepsilon$. Thus $\lim a_k = 0$.

(5) a Base case: When $m=0$, $1=\frac{-a}{1-a}$

Inductive step: Suppose $1+a+\_\_+a^{m-1}=\frac{1-a^m}{1-a}$.

Then $1+a+\dots+a^{m-1}+a^m=\frac{1-a^m}{1-a}+a^m$

$=\frac{1-a^m+(1-a)a^m}{1-a}=\frac{1-a^m-a^m-a^{m+1}}{1-a}=\frac{1-a^{m+1}}{1-a}$,

which completes the proof.

(b) By the hint and part (a),

$$\sum_{i=n}^{m-1} a^i=\sum_{i=0}^{m-1} a^i-\sum_{i=0}^{n-1} a^i=\frac{1-a^m}{1-a}-\frac{1-a^n}{1-a}=\frac{a^n-a^m}{1-a}$$

(c) Note that, for $m>n$,

$$|s_m-s_n|=|s_m-s_{m-1}+s_{m-1}-s_{m-2}+\dots+s_{n+1}-s_n|$$

$$\stackrel{(c)}{\le}|s_m-s_{m-1}|+|s_{m-1}-s_{m-2}|+\_\_+|s_{n+1}-s_n|$$

$$\le 4^{-(m-1)}+4^{-(m-2)}+\dots+4^{-n}$$

$$\stackrel{(b)}{=}\frac{\left(\frac{1}{4}\right)^n-\left(\frac{1}{4}\right)^m}{\frac{3}{4}}$$

$$\le\frac{4}{3}\left(\frac{1}{4}\right)^n.$$

Furthermore, for all $\varepsilon>0$,

$$\frac{4}{3}\left(\frac{1}{4}\right)^n<\varepsilon \iff \left(\frac{1}{4}\right)^n<\frac{3\varepsilon}{4}\iff n\log\left(\frac{1}{4}\right)<\log\left(\frac{3\varepsilon}{4}\right)$$

$$\iff n>\log\left(\frac{3\varepsilon}{4}\right)\Big/\log\left(\frac{1}{4}\right).$$

Let $\varepsilon>0$. Define $N=\frac{\log\left(\frac{3\varepsilon}{4}\right)}{\log\left(\frac{1}{4}\right)}$. Then $m,n>N$ ensures $|s_m-s_n|<\varepsilon$.

Therefore s_n is Cauchy

(d) Yes. The sequence s_n converges since all Cauchy sequences are convergent.

(6) (a) By definition $s_{n+1} = s_n + \frac{d_{n+1}}{10^{n+1}}$. Since $d_{n+1} \geq 0$, $s_{n+1} \geq s_n$.

(b) Taking $a = \frac{1}{10}$ in Q5(a) gives

$$1 + \frac{1}{10} + \frac{1}{10^2} + \dots + \frac{1}{10^n} = \frac{1 - (1/10)^{n+1}}{9/10}$$

$$\Leftrightarrow 9 + \frac{9}{10} + \frac{9}{10^2} + \dots + \frac{9}{10^n} = 10 - (1/10)^n$$

$$\Leftrightarrow \quad \frac{9}{10} + \frac{9}{10^2} + \dots + \frac{9}{10^n} = 1 - \left(\frac{1}{10}\right)^n$$

(c) Since $s_n = K + \frac{d_1}{10} + \frac{d_2}{10^2} + \dots + \frac{d_n}{10^n}$ and $d_i \leq 9$ for all $i = 1, \dots, n$,

$s_n \leq K + \frac{9}{10} + \frac{9}{10^2} + \dots + \frac{9}{10^n} = K + 1 - \frac{1}{10^n} \leq K + 1$.

Therefore s_n is bounded above. Since $s_n \geq 0$, it is also bounded below, hence bounded.

(d) Let $s_n = .\overbrace{99\ldots9}^{n \text{ times}}$. Then $s_n = 1 - \frac{1}{10^{n+1}}$.

Since $\lim_{n\to\infty} \frac{1}{10^n} = 0$, $\lim_{n\to\infty} \frac{1}{10^{n+1}} = \lim_{n\to\infty} \frac{1}{10^n}\frac{1}{10} = 0$,

hence $\lim_{n\to\infty} 1 - \frac{1}{10^{n+1}} = 0$. Thus,

$.\overline{9} = \lim_{n\to\infty} s_n = 1$.

⑦ Define $s_n = \sum_{k=1}^{n} r^k$.

(a) $\sum_{k=1}^{\infty} r^k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1-r^{n+1}}{1-r} \overset{|r|<1}{=} \frac{1-0}{1-r} = \frac{1}{1-r}$.

(b) By the corollary, if $\sum_{k=1}^{\infty} r^k$ converges, then $\lim_{k\to\infty} r^k = 0$. Thus, if we can show $\lim_{k\to\infty} r^k \neq 0$, we must have $\sum_{k=1}^{\infty} r^k$ doesn't converge.

If $r>1$, $\lim_{k\to\infty} r^k = +\infty$ and if $r<-1$ $\lim_{k\to\infty} r^k$ does not exist. Thus, if $|r| \geq 1$, $\lim_{k\to\infty} r^k \neq 0$.

If $r=1$, $\lim_{k\to\infty} r^k = 1$ and if $r=-1$, $\lim_{k\to\infty} r^k$ D.N.E..

8. First, note that $\liminf \sigma_n \le \limsup \sigma_n$ by definition of liminf and limsup.

We now show $\liminf s_n \le \liminf \sigma_n$ by first proving the hint.

Note that if $n > M > N$

$$\begin{aligned}
\sigma_n &= \frac{1}{n}(s_1 + s_2 + \dots + s_n) \\
&= \frac{1}{n}(s_1 + s_2 + \dots + s_N + s_{N+1} + \dots + s_M + \dots + s_n) \\
&\ge \frac{1}{n}(s_{N+1} + \dots + s_M + \dots + s_n) \quad s_i \ge 0 \text{ for all } i \\
&\ge \frac{1}{n}(n-N) \inf\{s_n : n > N\} \\
&= \left(1 - \frac{N}{n}\right) \inf\{s_n : n > N\} \\
&\ge \left(1 - \frac{N}{M}\right) \inf\{s_n : n > N\} \quad \text{since } n > M
\end{aligned}$$

since for $i > N$, $s_i > \inf\{s_n : n > N\}$ and there are $(n-N)$ elements in the sum

Therefore $\left(1-\frac{N}{M}\right)\inf\{s_n : n > N\}$ is a lower bound for the set $\{\sigma_n : n > M\}$.

Hence $\underbrace{\inf\{\sigma_n : n > M\}}_{B_M} \ge \left(1 - \frac{N}{M}\right) \underbrace{\inf\{s_n : n > N\}}_{b_N}$.

First suppose N is fixed. Since $B_M \ge \left(1-\frac{N}{M}\right) b_N$ for all $M > N$, sending $M \to +\infty$ gives $\liminf \sigma_n = \lim_{M\to\infty} B_M \ge b_N$.

Now, sending $N \to +\infty$ gives $\liminf \sigma_n \ge \lim_{N\to\infty} b_N = \liminf s_n$, which proves the first inequality.

Now we show $\limsup \sigma_n \le \limsup s_n$ by proving the other hint.

Note that if $n > M > N$,

$$\sigma_n = \frac{1}{n}(s_1 + s_2 + \dots + s_N + s_{N+1} + \dots + s_M + \dots + s_n)$$

$$= \frac{1}{n}(s_1 + s_2 + \dots + s_N) + \frac{1}{n}(s_{N+1} + \dots + s_M + \dots + s_n)$$

since for $i > N$ $s_i \le \sup\{s_n : n > N\}$ and there are $(n-N)$ elements in the second sum

$$\le \frac{1}{n}(s_1 + s_2 + \dots + s_N) + \frac{1}{n}(n-N)\sup\{s_n : n > N\}$$

$$\le \frac{1}{n}(s_1 + s_2 + \dots + s_N) + \sup\{s_n : n > N\} \quad \frac{1}{n}(n-N) \le 1$$

$$\overset{n>M}{\le} \frac{1}{M}(s_1 + s_2 + \dots + s_N) + \sup\{s_n : n > N\}$$

Thus $\underbrace{\sup\{\sigma_n : n > M\}}_{A_M} \le \frac{1}{M}(s_1 + s_2 + \dots + s_N) + \underbrace{\sup\{s_n : n > N\}}_{a_N}$.

Sending $M \to +\infty$ for fixed N gives,

$\limsup \sigma_n = \lim_{M\to\infty} A_M \le 0 + a_N$.

Then sending $N \to \infty$ gives

$\limsup \sigma_N \le \lim_{N\to\infty} a_N = \limsup s_n$,

which completes the proof.

(b) If $\lim s_n$ exists, then $\limsup s_n = \liminf s_n$. Hence, by part (a), $\limsup \sigma_n = \liminf \sigma_n$. Therefore $\lim \sigma_n$ exists.

(c) Consider $s_n = (-1)^{n+1}$, so $\lim s_n$ doesn't exist. Then $\sigma_n = \begin{cases} \frac{1}{n} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even,} \end{cases}$

so $\lim \sigma_n = 0$.

9

(a) First, note that, for any $N \in \mathbb{N}$, if M_s is an upper bound for $\{s_n : n>N\}$ and M_t is an upper bound for $\{t_n : n>N\}$, then $M_s + M_t$ is an upper bound for $\{s_n + t_n : n > N\}$. Consequently, $\forall\, N \in \mathbb{N}$,

$$(*)\ \sup\{s_n + t_n : n>N\} \le \sup\{s_n : n>N\} + \sup\{t_n : n>N\}$$

Recall that

$$\limsup_{n\to\infty}(s_n + t_n) = \lim_{N\to\infty} \overbrace{\sup\{s_n + t_n : n > N\}}^{x_N}$$

$$\limsup_{n\to\infty} s_n = \lim_{N\to\infty} \underbrace{\sup\{s_n : n>N\}}_{y_N}$$

$$\limsup_{n\to\infty} t_n = \lim_{N\to\infty} \underbrace{\sup\{s_n : n>N\}}_{z_N}$$

We have $x_N \le y_N + z_N$ for all $N \in \mathbb{N}$. Furthermore, since s_n and t_n are bounded sequences, so are x_N, y_N, z_N. Since bounded monotone sequences converge, the limit of sum is sum of limits:

$$\lim_{N\to\infty} y_N + \lim_{N\to\infty} z_N = \lim_{N\to\infty} y_N + z_N$$

Hint

$$\geq \lim_{N\to\infty} x_N.$$

This completes the proof.

(b) Let $s_n = (-1)^n$, $t_n = (-1)^{n+1}$. Then $s_n + t_n = 0$. Thus,

$$\limsup_{n\to\infty} s_n + t_n = \lim_{n\to\infty} s_n + t_n = 0$$

$$\limsup_{n\to\infty} s_n = \lim_{N\to\infty} \sup\{(-1)^n : n > N\} = \lim_{N\to\infty} 1 = 1$$

$$\limsup_{n\to\infty} t_n = \lim_{N\to\infty} \sup\{(-1)^{n+1} : n > N\} = \lim_{N\to\infty} 1 = 1$$

Since $0 < 2$, this gives the result.

10 a) Claim: $S = \{0\} \cup \{\frac{1}{\ell} : \ell \in \mathbb{N}\}$

Since $s_{n_k} = \frac{1}{k}$ is a subsequence, $0 \in S$. Since $s_{n_k} = \frac{1}{\ell}$ is a subsequence for all $\ell \in \mathbb{N}$, $\frac{1}{\ell} \in S$.

It remains to show no other real number or $\pm\infty$ belongs to S.

Neither $+\infty$ nor $-\infty$ belong to S, since the sequence is bounded.

Suppose $a \in S$ for some $a \in \mathbb{R}$. By the main subsequences theorem, it suffices to show $\exists\ \varepsilon_0 > 0$ so that $|a - s_n| \geq \varepsilon_0$ for all n.

$\boxed{\text{If } a > 1}$, then $|a - s_n| \geq |a - 1| =: \varepsilon_0\ \forall n$

$\boxed{\text{If } a < 0}$, then $|a - s_n| > |a| =: \varepsilon_0\ \forall n$.

$\boxed{\text{If } \frac{1}{\ell} > a > \frac{1}{\ell+1} \text{ for some } \ell \in \mathbb{N}}$ then $|a - s_n| \geq \min\{|a - \frac{1}{\ell}|, |a - \frac{1}{\ell+1}|\} =: \varepsilon_0\ \forall n$

This completes the proof.

(b) $\limsup s_n = \max(S) = 1$
$\liminf s_n = \min(S) = 0$

(11) (a) If s_{n_k} is bounded, by Bolzano-Weierstrass, s_{n_k} must have a convergent subsequence $s_{n_{k_\ell}}$. Since $s_{n_{k_\ell}}$ is also a subsequence of s_n, s_n has a convergent subsequence.

(b) Suppose $|s_n|$ does not diverge to $+\infty$. Then $\exists M > 0$ s.t. $\forall N, \exists n > N$ for which $|s_n| \leq M$. Since $|s_n| \geq 0$ for all $n \in \mathbb{N}$, this implies there exist infinitely many $n \in \mathbb{N}$ for which $0 \leq |s_n| \leq M$. Consequently, there exists a subsequence s_{n_k} for which $0 \leq |s_{n_k}| \leq M$ $\forall k \in \mathbb{N}$. Therefore s_{n_k} is a bounded sequence, so by part (a), s_n must have a convergent subsequence.

12 (a) If $\lim s_n = s$, then all subsequences of s_n also converge to s. Hence every subsequence s_{n_k} has a further subsequence $s_{n_{k_\ell}} = s_{n_k}$ that converges to s.

(b) Suppose $\lim s_n \neq s$. Then,
$\exists\, \varepsilon > 0$ s.t. $\forall N$, $\exists\, n > N$ s.t. $|s_n - s| \geq \varepsilon$

First, taking $N = 1$, we have $\exists\, n_1 > 1$ s.t. $|s_{n_1} - s| \geq \varepsilon$. Suppose we have chosen n_{k-1}. Taking $N = n_{k-1}$, we see that $\exists\, n_k > n_{k-1}$ s.t. $|s_{n_k} - s| \geq \varepsilon$.

Therefore there exists a subsequence s_{n_k} s.t. $|s_{n_k} - s| \geq \varepsilon$ $\forall k$. Since s_{n_k} is always at least distance ε from s, no further subsequence of s_{n_k} can converge to s.