Math 117: Homework 7

Due Friday, June 6th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1

Suppose $S \subseteq \mathbb{R}$ is a bounded set and f is a bounded function on S. Define $M = \sup\{f(s) : s \in S\}$.

(a) Use the definition of the supremum to prove that there exists a sequence s_n of elements in S so that $\lim_{n\to+\infty} f(s_n) = M$.

(b) Now, use the result from part (a) to prove that there exists a *convergent* sequence t_k of elements in S so that $\lim_{k\to+\infty} f(t_k) = M$.

Question 2*

Consider the set $S = \left\{\sqrt{5}r : r \in \mathbb{Q}\right\}$. You may assume that $\sqrt{5}$ is an irrational number.

(a) Prove that S is dense in $\mathbb{R}$ by showing that, for all $a, b \in \mathbb{R}$ with $a < b$, there exists $s \in S$ satisfying $a < s < b$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that if $f(s) = \pi$ for all $s \in S$, then $f(x) = \pi$ for all $x \in \mathbb{R}$.

(c) Let f and g be continuous real-values functions defined on $\mathbb{R}$ such that $f(s) = g(s)$ for each $s \in S$. Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$. (Hint: define a new function and apply part (b).)

As a consequence, you see that if two continuous functions on $\mathbb{R}$ are equal on a dense subset of $\mathbb{R}$, they must actually be equal everywhere.

Question 3*

(a) Consider $S \subseteq \mathbb{R}$ bounded, and suppose $f : S \to \mathbb{R}$ is uniformly continuous. Prove that f is bounded on S.

(b) Does the conclusion of part (a) still hold if f is merely continuous? Justify your answer with a proof or counterexample.

Question 4*

Fix $x_0, x_1 \in \mathbb{R}$ with $x_0 < x_1$. Prove that $f : (x_0, x_1) \to \mathbb{R}$ is continuous if and only if for all $a \in (x_0, x_1)$,

$$\lim_{x\to a} f(x) = f(a).$$

Question 5

Suppose f_n is a sequence of uniformly continuous functions on $S \subseteq \mathbb{R}$, and suppose there exists $f : S \to \mathbb{R}$ so that $f_n \to f$ uniformly on S. Prove that f is uniformly continuous on S.

Practice Final Exam

Question 1

Given $S \subset \mathbb{R}$, a function $f : S \to \mathbb{R}$ is *increasing* if, for all $x, y \in S$,

$$x \leq y \iff f(x) \leq f(y).$$

Inspired by HW7 Q4, we define a function $f : \mathbb{R} \to \mathbb{R}$ to be *right continuous* if, for all $a \in \mathbb{R}$,

$$\lim_{x \to a^+} f(x) = f(a).$$

(a) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is right continuous but not continuous. Justify your example with a proof.

(b) Suppose f_n is a sequence of continuous, increasing functions. Suppose that, for each $x \in \mathbb{R}$, $f_n(x)$ is a decreasing sequence and f_n converges pointwise to f. Prove that f is right continuous.

This problem shows that, while the pointwise limit of continuous functions is not generally continuous, as long as suitable monotonicity hypotheses hold, the limit will be right continuous.

Question 2

Suppose f_n is a sequence of continuous functions on a set $[a, b]$ and there exists $f : [a, b] \to \mathbb{R}$ so that $f_n \to f$ uniformly on $[a, b]$. Prove that, for any sequence $x_n \in [a, b]$ with $\lim_{n \to +\infty} x_n = x$, we have $\lim_{n \to +\infty} f_n(x_n) = f(x)$.

Question 3

Determine whether the following statements are true or false. If they are true, prove them. If they are false, provide a counterexample and justify your counterexample.

(i) If $\limsup_{n \to +\infty} x_n \leq \liminf_{n \to +\infty} y_n$, then $x_n \geq y_n$ for at most finitely many n.

(ii) If $\limsup_{n \to +\infty} x_n < \liminf_{n \to +\infty} y_n$, then $x_n \geq y_n$ for at most finitely many n.

Question 4

Parts a and b are not related

(a) Fix $a \in \mathbb{R}$ and consider the function

$$g_a(x) = \begin{cases} \frac{1}{x} & \text{for } x \neq 0, \\ a & \text{for } x = 0. \end{cases}$$

Prove that g_a is not continuous.

(b) Suppose f is a continuous function on $\mathbb{R}$ and $f(a)f(b) < 0$ for some $a, b \in \mathbb{R}$. Prove there exists x between a and b such that $f(x) = 0$.