Homework 7 Solutions

(1)

(a) Define a sequence s_n as follows: since M is the least upper bound for $\{f(x) : x \in S\}$, $\forall n \in \mathbb{N}$, $\exists x_n \in S$ s.t. $M - \frac{1}{n} < f(x_n) \leq M$. Thus, by the Squeeze Lemma, $\lim_{n \to \infty} f(x_n) = M$.

(b) Since $x_n \in S$ and S is bounded, x_n is a bounded sequence. By Bolzano-Weierstrass, it has a convergent subsequence x_{n_k}. Since $f(x_{n_k})$ is a subsequence of $f(x_n)$, $\lim_{k \to \infty} f(x_{n_k}) = M$. Thus, the result holds with $t_k = x_{n_k}$.

(2)

(a) Fix $a, b \in \mathbb{R}$ with $a<b$. Then we also have $\frac{a}{\sqrt{5}} < \frac{b}{\sqrt{5}}$. By density of $\mathbb{Q}$ in $\mathbb{R}$, $\exists$ $r \in \mathbb{Q}$ s.t.

$$\frac{a}{\sqrt{5}} < r < \frac{b}{\sqrt{5}}.$$

Thus, $a < \sqrt{5}r < b$. Since $\sqrt{5}r \in S$, this shows the result.

(b) Fix $x \in \mathbb{R}$. By part (a), $\forall n \in \mathbb{N}$, $\exists s_n \in S$ s.t. $x < s_n < x + \frac{1}{n}$. Then $\lim_{n\to\infty} s_n = x$, by the Squeeze Lemma. Since f is continuous, $\lim_{n \to \infty} f(s_n) = f(x)$. Finally, since $f(s_n) = \pi \ \forall n \in \mathbb{N}$, we see that $f(x) = \pi$.

(c) Let $h(x) = f(x) - g(x) + \pi$. Since f and g are continuous and constant functions are continuous, using that the sum and product of continuous functions is continuous, we have

$$h(x) = \underbrace{f(x) + \overbrace{(-g(x))}^{\text{cts, since product of } g \text{ and } -1} + \pi}_{\text{sum of cts is cts}}$$

is continuous.

Furthermore, by definition, $h(s) = \pi$ for all $s \in S$. By part (a), we see $h(x) = \pi$ for all $x \in \mathbb{R}$. This shows $f(x) - g(x) = 0 \ \forall x \in \mathbb{R}$, thus $f(x) = g(x)$ for all $x \in \mathbb{R}$.

3 a Assume, for the sake of contradiction, that f is not bounded on S. Then, $\nexists\, M>0$ s.t. $|f(x)| \leq M \; \forall\, x \in S$. Hence, $\forall\, n \in \mathbb{N}$, $\exists\, x_n \in S$ s.t. $|f(x_n)| > n$.

Since $S \subseteq \mathbb{R}$ is bounded, x_n is bounded. By Bolzano-Weierstrass, it has a convergent subsequence x_{n_k}. Since x_{n_k} is Cauchy and f is unif cts, $f(x_{n_k})$ is Cauchy, hence bounded. This contradicts the fact that $|f(x_{n_k})| > n_k \geq k \; \forall\, k \in \mathbb{N}$.

Thus, f must be bounded on S.

b No. Consider $S = (0,1)$ and $f(x) = \frac{1}{x}$.

④ Suppose f is cts on (x_0, x_1).
Fix $a \in (x_0, x_1)$. By cty, for any sequence $x_n \in (x_0, x_1) \setminus \{a\}$ satisfying $x_n \to a$, we have $f(x_n) \to f(a)$.
Thus, $\lim_{x \to a} f(x) = f(a)$

Now suppose $\overbrace{\lim_{x \to a} f(x) = f(a)}^{(*)}$ for all $a \in (x_0, x_1)$.

Fix $a \in (x_0, x_1)$ and $x_n \in (x_0, x_1)$ satisfying $x_n \to a$. Every subsequence x_{n_k} has a further subsequence $x_{n_{k_\ell}}$ satisfying either

case 1: $x_{n_{k_\ell}} = a \quad \forall \ell \in \mathbb{N}$
case 2: $x_{n_{k_\ell}} \neq a \quad \forall \ell \in \mathbb{N}$

In the first case, $\lim_{\ell \to \infty} f(x_{n_{k_\ell}}) = a$.
In the second case, our hypothesis $(*)$ ensures $\lim_{\ell \to \infty} f(x_{n_{k_\ell}}) = a$.

Thus, every subsequence of $f(x_n)$ has a further subsequence that converges to $f(a)$. This shows $f(x_n)$ converges to $f(a)$. Thus f is cts at a. Since $a \in (x_0, x_1)$ was arbitrary, f is cts on (x_0, x_1).

⑤ Fix $\varepsilon > 0$. We must show $\exists\, \delta > 0$ s.t. $x, y \in S$ and $|x-y| < \delta$ ensures $|f(x) - f(y)| < \varepsilon$.

Since $f_n \to f$ uniformly on S, $\exists\, N$ s.t. $n > N$ ensures $|f_n(x) - f(x)| < \varepsilon/3$, $\forall x \in S$. Choose $N_1 \in \mathbb{N}$ s.t. $N_1 > N$. Since f_{N_1} is unif cts on (a,b), $\exists\, \delta > 0$ s.t. $x, y \in S$ and $|x-y| < \delta$ ensures $|f_{N_1}(x) - f_{N_1}(y)| < \varepsilon$.

Thus, $x, y \in S$ and $|x-y| < \delta$ ensures

$$
\begin{aligned}
|f(x)-f(y)| &\leq |f(x)-f_{N_1}(x)| + |f_{N_1}(x)-f_{N_1}(y)| \\
&\quad + |f_{N_1}(y)-f(y)| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}
$$

This gives the result.

Solutions to Practice Final Exam

① ⓐ Consider $f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$,

If $a \geq 0$, for any $x_n \in (a, +\infty)$ satisfying $x_n \to a$, $f(x_n) = 0 \ \forall n \in \mathbb{N}$, so $f(x_n) \to 0$. Thus $\lim_{x \to a^+} f(x) = f(a)$.

If $a < 0$, for any $x_n \in (a, +\infty)$ satisfying $x_n \to a$, for n sufficiently large, $x_n < a + \frac{|a|}{2} = \frac{a}{2} < 0$. Thus, $f(x_n) \to 1$. This shows $\lim_{x \to a^+} f(x) = f(a)$

We have thus shown f is right cts. To see that f is not continuous, note that $-\frac{1}{n} \to 0$, but $f(-\frac{1}{n}) = 1$ does not converge to $f(0) = 0$.

(b) Note that $a \le b \Leftrightarrow f_n(a) \le f_n(b)\ \forall n$

Thus $a \le b \Leftrightarrow f(a) \le f(b)$,

so f is also increasing.

Fix $a \in \mathbb{R}$. We must show $\lim_{x \to a^+} f(x) = f(a)$.

Fix $x_n \in (a, +\infty)$ satisfying $x_n \to a$.

It suffices to show $f(x_n) \to f(a)$

First, we will prove the result under the additional hypothesis that x_n is decreasing. Fix $\varepsilon > 0$.

$$|f(x_n) - f(a)| = f(x_n) - f(a) \quad \text{since } f \text{ increasing}$$

$$\le f_N(x_n) - f(a) \quad \text{for any } N \in \mathbb{N}, \text{ since } f_N \text{ decreasing}$$

$$\le f_N(x_n) - f_N(a) + f_N(a) - f(a)$$

Choose N sufficiently large so that $f_N(a) - f(a) < \varepsilon/2$. Choose N_0 s.t. $n > N_0$ ensures $f_N(x_n) - f_N(a) < \frac{\varepsilon}{2}$.

Then $\forall\, n > N_0$,

$|f(x_n) - f(a)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

(2) Since $a \leq x_n \leq b \;\forall\, n \in \mathbb{N}$, $x \in [a,b]$.

Fix $\varepsilon > 0$ arbitrary. Choose N_0 s.t. $n > N_0$ ensures $|f_n(y) - f(y)| < \frac{\varepsilon}{2}$ $\forall\, y \in [a,b]$. Since the uniform limit of cts fns is cts, f is cts on $[a,b]$, so $\exists\; N_1$ s.t. $n > N_1$ ensures $|f(x_n) - f(x)| < \frac{\varepsilon}{2}$.

Then $n > \max\{N_0, N_1\}$ ensures

$$\begin{aligned} |f_n(x_n) - f(x)| &\leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

(3) (i) False - suppose $x_n = y_n = (1,1,1,\dots)$.

Then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 1$, so

$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} y_n$, but $x_n = y_n$ for all $n \in \mathbb{N}$.

(ii) True. Assume, for the sake of contradiction that there exist infinitely many n so that $x_n \geq y_n$. Then, for any $N \in \mathbb{N}$, there exists $k > N$ so that $x_k \geq y_k$. Thus, $\forall N \in \mathbb{N}$,

$$\underbrace{\sup\{x_n : n > N\}}_{a_N} \geq x_k \geq y_k \geq \underbrace{\inf\{y_n : n > N\}}_{b_N}.$$

Since the limits of a_N and b_N exist, this shows

$\limsup_{n\to\infty} x_n = \lim_{N\to\infty} a_N \geq \lim_{N\to\infty} b_N = \liminf_{n\to\infty} y_n$.

This is a contradiction.

④

ⓐ Consider the sequence $\frac{1}{n} \to 0$.
Then $\lim_{n\to\infty} g_a(\frac{1}{n}) = \lim_{n\to\infty} n = +\infty$

$\neq a = g_a(0)$. This shows $g_a(x)$ is not continuous at $x=0$, hence not continuous.

ⓑ WLOG, suppose $a \leq b$.

Since $f(a)f(b) < 0$, $f(a)$ and $f(b)$ must be of opposite sign. Thus, by the intermediate value theorem, $\exists\ x \in [a,b]$ s.t. $f(x) = 0$.