Lecture 10
CS 117, S25
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Announcements:

- Makeup lecture on Friday, May 11am - 12:15pm

Recall:

Def: Given a sequence $s_n, n \in \mathbb{N}$, and a strictly increasing sequence n_k of natural numbers, a sequence of the form s_{n_k} is a subsequence of s_n.

Lemma: For any strictly increasing sequence of natural numbers n_k, we have $n_k \geq k \ \forall k \in \mathbb{N}$.

Def: A subsequential limit of a sequence s_n is a real number or symbol $\pm\infty$ that is the limit of some subsequence of s_n.

Thm: If a sequence s_n converges to $s \in \mathbb{R}$, then every subsequence also converges to s.

Thm (main subsequence theorem)

Let s_n be a sequence.

(a) For any $t \in \mathbb{R}$

[t is a subsequential limit of s_n]

$\Updownarrow$

[the set $\{n : |s_n - t| < \varepsilon\}$ is infinite, for all $\varepsilon > 0$]

$\Updownarrow$ ← - another way of writing same thing

[$\forall \varepsilon > 0$, $|\{n : |s_n - t| < \varepsilon\}| = +\infty$]

(b) $+\infty$ is a subsequential limit

$\iff$ s_n is unbounded above

(c) $-\infty$ is a subsequential limit

$\iff$ s_n is unbounded below

Lemma: ~~If~~ s_n is unbounded above, ~~then~~ iff $\forall\, M>0$, $|\{n : s_n > M\}| = +\infty$.

Question:
is (a) $\Longleftrightarrow$ [the set $\{s_n : |s_n - t| < \varepsilon\}$ is infinite for all $\varepsilon > 0$]

No.

Ex: $s_n = (1, 1, 1, \ldots)$, $t = 1$

$\forall \varepsilon > 0$, $\{n : |s_n - t| < \varepsilon\} = \mathbb{N}$, so
$|\{n : |s_n - t| < \varepsilon\}| = +\infty$

$\forall \varepsilon > 0$, $\{s_n : |s_n - t| < \varepsilon\} = \{s_n : n \in \mathbb{N}\}$
$= 1$, so
$|\{s_n : |s_n - t| < \varepsilon\}| = 1 < +\infty$.

OTOH "$\Leftarrow$" is true.

Another Question: We have shown

s_n unbdd above

$\Rightarrow \forall M>0, |\{n : s_n > M\}| = +\infty$

Is "$\Leftarrow$" true? Yes.

Pf of Main Subsequences Thm:

Fix a sequence s_n.

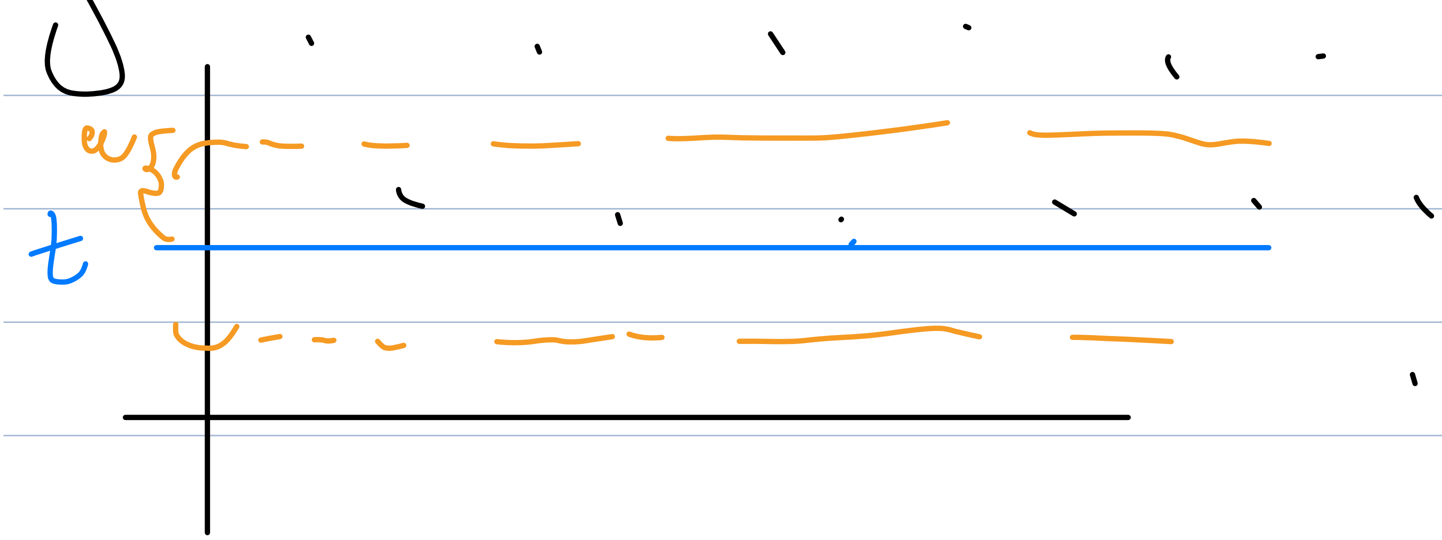
(a) Fix $t \in \mathbb{R}$.

$t-\varepsilon < s_n < t+\varepsilon$

$\Updownarrow$

Suppose $\forall \varepsilon > 0$, $|\{n : |s_n - t| < \varepsilon\}| = +\infty$.

WTS t is a subsequential limit of s_n.



Goal: show there exists a subsequence s_{n_k} of s_n satisfying

$$t-\frac{1}{k} \le s_{n_k} \le t+\frac{1}{k} \quad \forall k \in \mathbb{N}.$$

If we could find such a subsequence, the Squeeze Lemma would then ensure $\lim_{k\to\infty} s_{n_k} = t$.

We will construct the subsequence inductively.

For $k=1$, since $|\{n : t-1 < s_n < t+1\}| = +\infty$, the set is nonempty. Choose $\tilde{n} \in \{n : t-1 < s_n < t+1\}$. Let $n_1 := \tilde{n}$, so $s_{n_1} = s_{\tilde{n}}$.

For $k=2$, since $|\{n : t-\frac{1}{2} < s_n < t+\frac{1}{2}\}| = +\infty$

Thus, there exists

$\tilde{\tilde{n}} \in \{n : t - \frac{1}{2} < s_n < t + \frac{1}{2}\}$ and $\tilde{\tilde{n}} > \tilde{n}$

Let $n_2 = \tilde{\tilde{n}}$, so $s_{n_2} = s_{\tilde{\tilde{n}}}$.

Suppose we have constructed all elements in our subsequence up to $s_{n_{k-1}}$. Since $|\{n : t - \frac{1}{k} < s_n < t + \frac{1}{k}\}| = +\infty$,
$\exists\, m \in \{n : t - \frac{1}{k} < s_n < t + \frac{1}{k}\}$ s.t. $m > n_{k-1}$.
Let $n_k = m$, so $s_{n_k} = s_m$.

Therefore $\lim_{k \to \infty} s_{n_k} = t$, so t is a subsequential limit.

Now, suppose t is a subsequential limit. WTS $\forall \varepsilon > 0$, $|\{n : |s_n - t| < \varepsilon\}| = +\infty$.

Fix $\varepsilon > 0$ arbitrary. Since t is a subsequential limit, there exists a subsequence s_{n_k} that converges to t. Thus there exists K s.t. $k \geq K$ ensures $|s_{n_k} - t| < \varepsilon$. In other words,

$$\{n_k : k \geq K\} \subseteq \{n : |s_n - t| < \varepsilon\}.$$

Since the set on the LHS is infinite, so is the set on the RHS. is infinite.

(b)

Suppose s_n is unbounded above. WTS $\exists$ a subsequence s_{n_k} s.t. $\lim_{k \to \infty} s_{n_k} = +\infty$.

(b) Suppose [s_n is unbounded above].
By the lemma, for all $M>0$, $\{n : s_n > M\}$ is infinite. Hence, we may construct a subsequence as follows.
Choose n_1 so that $s_{n_1} > 1$.
Choose n_2 so that $s_{n_2} > 2$ and $n_2 > n_1$.
$\vdots$
Choose n_k so that $s_{n_k} > k$ and $n_k > n_{k-1}$.

Fix $\tilde{m} > 0$. For $k > \tilde{m}$, $s_{n_k} > k > \tilde{m}$.
Since $\tilde{m}$ was arbitrary, $\lim_{k\to\infty} s_{n_k} = +\infty$.
Thus $+\infty$ is a subsequential limit.

Suppose [$+\infty$ is a subsequential limit].
Assume, for the sake of contradiction, that s_n is bounded above, that is there exists $M>0$ s.t. $s_n \leq M$ for all $n \in \mathbb{N}$. Take s_{n_k} s.t. $\lim_{k\to\infty} s_{n_k} = +\infty$.
Then $s_{n_k} \leq M$ for all $k \in \mathbb{N}$. This is a contradiction.

(c) Note that
[s_n is unbounded below]
$\Downarrow$

[$-s_n$ is unbounded above]

$\Updownarrow$ (b)

[$+\infty$ is a subsequential limit of $-s_n$]

$\Updownarrow$

[$-\infty$ is a subsequential limit of s_n]

$\square$