Lecture 11

CS 117, S25

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Recall :)

Thm (main subsequence theorem)

Let s_n be a sequence.

(a) For any $t \in \mathbb{R}$

[t is a subsequential limit of s_n]

$\Updownarrow$

[the set $\{n : |s_n - t| < \varepsilon\}$ is infinite, for all $\varepsilon > 0$]

$\Updownarrow$ ←-- another way of writing same thing

[$\forall \varepsilon > 0$, $|\{n : |s_n - t| < \varepsilon\}| = +\infty$]

(b) $+\infty$ is a subsequential limit

$\Longleftrightarrow$ s_n is unbounded above

(c) $-\infty$ is a subsequential limit

$\Longleftrightarrow$ s_n is unbounded below

Thm: Every sequence has a monotone subsequence.

Immediate consequence is...

Thm (Bolzano–Weierstrass): All bounded sequences have a convergent subsequence.

MAJOR THM 5

Pf (that every sequence has a monotone subsequence): Fix an arbitrary sequence s_n.

We will say that the n^{th} element of a sequence is dominant if it is greater than every element that follows, that is, if $s_n > s_m \ \forall\ m \geq n$.

$\boxed{\text{Case 1}}$ Suppose s_n has infinitely many dominant elements.

Define s_{n_k} to be the subsequence of all dominant elements.
Then $s_{n_k} > s_{n_{k+1}}$ $\forall k \in \mathbb{N}$,
so s_{n_k} is (strictly) decreasing, hence monotone.

$\boxed{\text{Case 2}}$ Suppose s_n has finitely many dominant elements.

- Choose n_1 so that s_{n_1} is beyond all the dominant elements in the sequence.
- Since s_{n_1} is not dominant, $\exists$ $m > n_1$ s.t. $s_m \geq s_{n_1}$.
  Let $n_2 := m$, so $s_{n_2} = s_m$.

- Suppose we have constructed all elements of our subsequence up to $s_{n_{k-1}}$ so that it is increasing. Since $s_{n_{k-1}}$ is not dominant, $\exists$ $\tilde{m} > n_{k-1}$ s.t. $s_{\tilde{m}} \geq s_{n_{k-1}}$. Let $n_k := \tilde{m}$, so $s_{n_k} = s_{\tilde{m}}$.

In this way, we have constructed a subsequence that is increasing, hence montone. $\square$

What is the connection between subsequences and limsup/liminf?

Downside: in general,

$a_N = \sup\{s_n : n > N\}$

$b_N = \inf\{s_n : n > N\}$ are not

subsequences.

Upside:

Thm: For any sequence s_n, $\limsup s_n$ and $\liminf s_n$ are the largest and smallest subsequential limits.

Ex: $s_n = (-1)^n$

The set of subsequential limits is $S = \{-1, 1\}$

$\limsup_{n\to\infty} s_n = 1$, $\liminf_{n\to\infty} s_n = -1$.

Pf:

$\boxed{\text{First, we will show that } \limsup s_n \text{ and } \liminf s_n \text{ are subsequential limits.}}$

We will begin with $\limsup s_n$.

$\boxed{\text{Case 1:}}$ $\limsup s_n = -\infty$.
Since $\liminf s_n \leq \limsup s_n$, we have $\liminf s_n = -\infty$. Thus, $\lim s_n = -\infty$, so $-\infty$ is a subsequential limit.

$\boxed{\text{Case 2:}}$ $\limsup s_n = +\infty$. Then $\lim_{N \to \infty} a_N = +\infty$. Hence, for any $m>0$, $\exists$ N_0 s.t. $N>N_0$ ensures $\sup\{s_n : n>N\} = a_N > m$.

If $\sup\{s_n : n>N\} = +\infty$ for some $N>N_0$, then $\{s_n : n>N\}$ is not bounded above, so the sequence must be unbounded above.

OTOH, if $\sup\{s_n : n>N\} \in \mathbb{R}$, since the supremum of a set is the <u>least</u> upper bound, we have that m is <u>not</u> an upper bound of $\{s_n : n>N\}$, so it can't be an upper bound of s_n. Since $m>0$ was arbitrary, s_n is unbounded above.

By Main Subseq Thm, $+\infty$ is a subseq. limit.

$\boxed{\text{Case 3.}}$ $\limsup s_n = t$ for $t \in \mathbb{R}$, so $\lim_{N\to\infty} a_N = t$. Fix $\varepsilon > 0$ arbitrary.

$t-\varepsilon < s_n < t+\varepsilon$

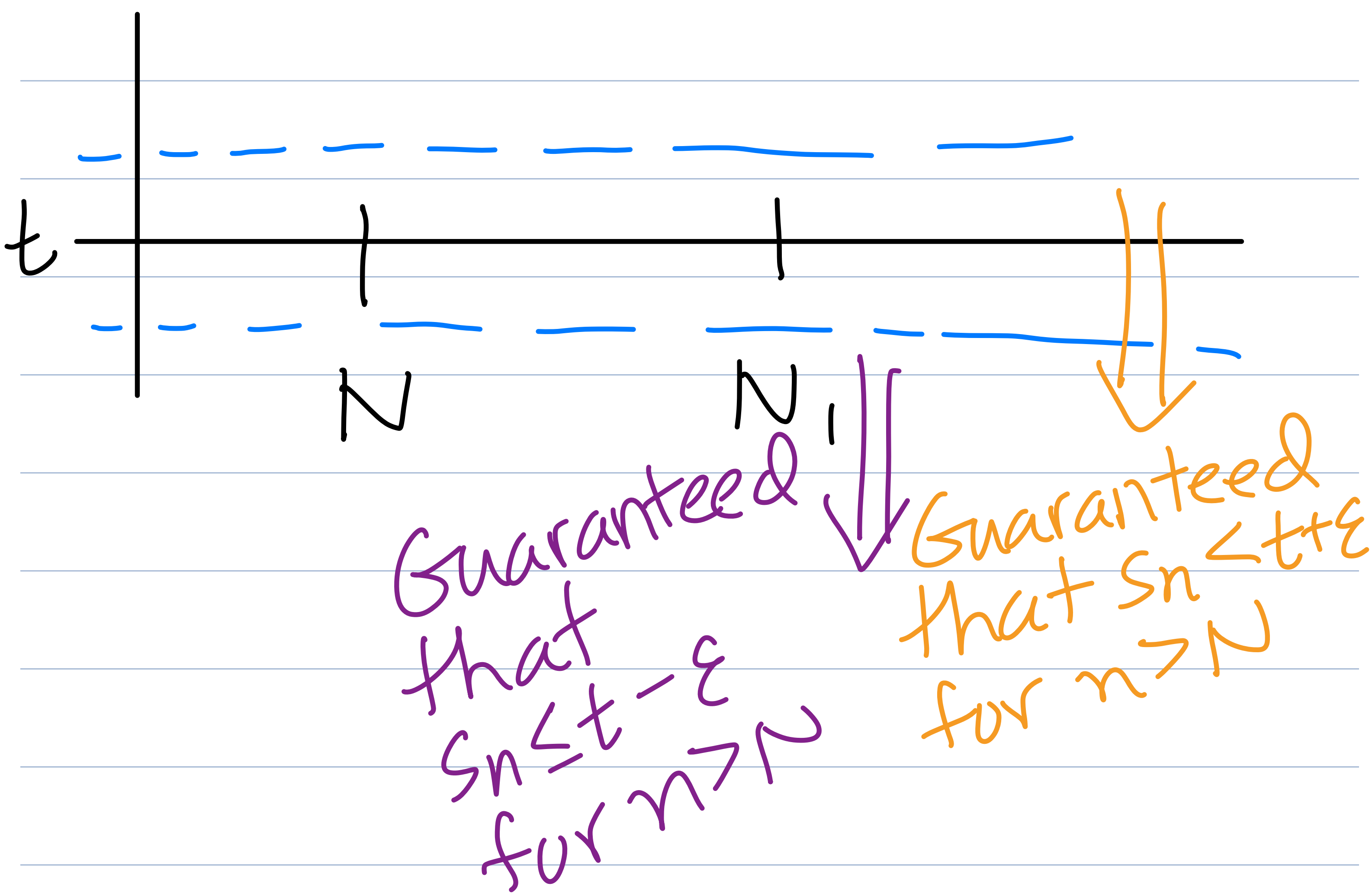
By Main Subseq Thm, it suffices to show $\{n : |s_n - t| < \varepsilon\}$ is infinite to conclude that t is a subseq limit.

Since $\lim_{N\to\infty} a_N = t$, $\exists$ N_0 s.t.

$N > N_0$ ensures $|a_N - t| < \varepsilon$
$\Rightarrow \sup\{s_n : n > N\} = a_N < t + \varepsilon$.

Thus, for any fixed $N > N_0$, for all $n > N$, $s_n < t + \varepsilon$

Assume, for the sake of contradiction that $\{n : t-\varepsilon < s_n < t+\varepsilon\}$ is finite. Since we know $n > N$ ensures $s_n < t+\varepsilon$, there must be $N_1 > N$ for which $s_n \le t-\varepsilon$ for all $n > N_1$.



Then $a_{N_1} = \sup\{s_n : n > N_1\} \leq t - \varepsilon$

But a_N is a decreasing sequence. This contradicts that $\lim_{N \to \infty} a_N = t$.

This completes the proof that t is a subsequential limit.

$\boxed{\text{Thus } \limsup s_n \text{ is always a subsequential limit of } s_n}$

Furthermore since $-\liminf s_n = \limsup -s_n$, by what we have already shown, $-\liminf s_n$ is a subsequential limit of $-s_n$.

$\boxed{\text{Thus, } \liminf s_n \text{ is a subsequential limit of } s_n.}$

It remains to show that $\limsup s_n$ and $\liminf s_n$ are the largest/smallest subsequential limits.

Suppose t is a subsequential limit, so that there is a subsequence s_{n_k} for which $\lim_{k\to\infty} s_{n_k} = t$.

Thus $t = \limsup_{k\to\infty} s_{n_k} = \lim_{K\to\infty} \sup\{s_{n_k} : k > K\}$

$t = \liminf_{k\to\infty} s_{n_k} = \lim_{K\to\infty} \inf\{s_{n_k} : k > K\}$