Lecture 12

CS 117, S25

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Recall:

Downside: in general,

$a_N = \sup\{s_n : n > N\}$

$b_N = \inf\{s_n : n > N\}$ are <u>not</u>.

subsequences.

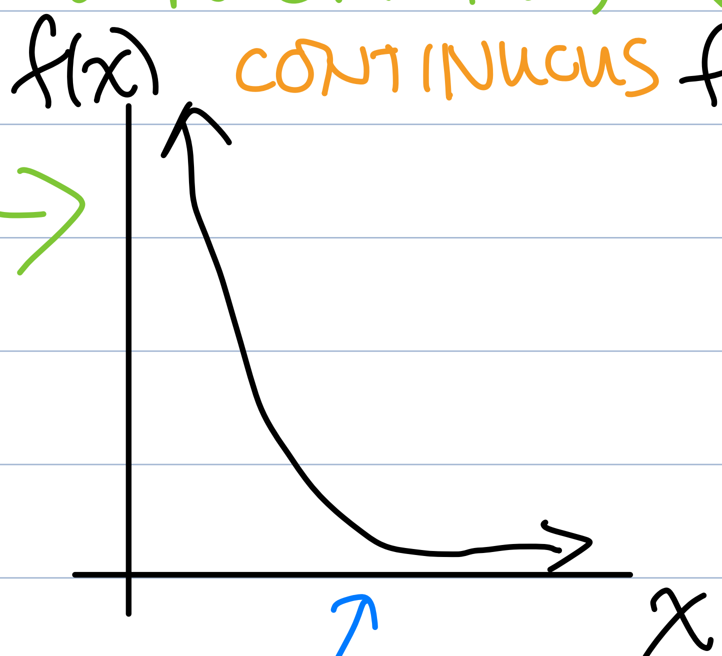
Upside:

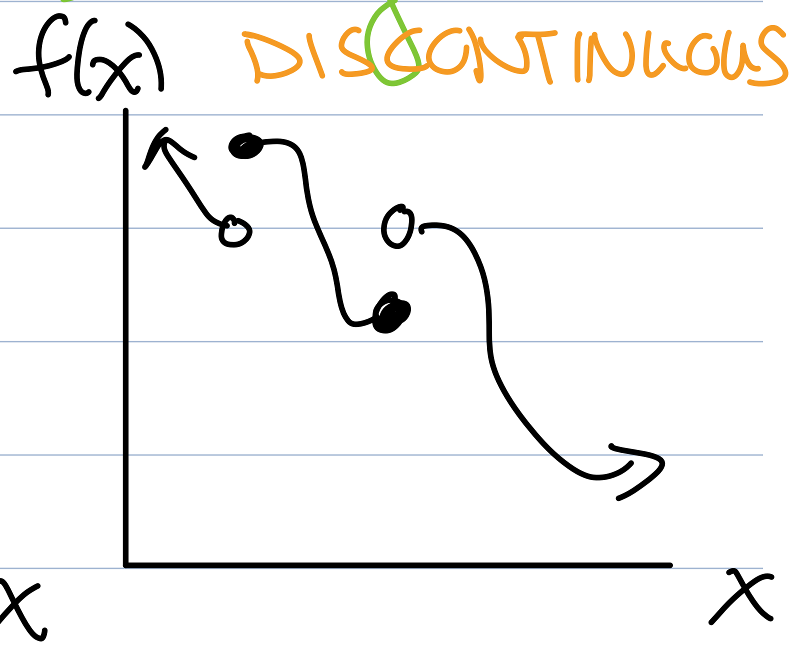
<u>Thm</u>: For any sequence s_n, $\limsup s_n$ and $\liminf s_n$ are the largest and smallest subsequential limits.

Now, apply the theory of sequences of real numbers to study continuous functions.

Intuitively, a function is continuous if it is an "unbroken curve" with "no holes"

range of f(x) is the set to which f(x) belongs





domain of f(x) is the set of x for which f(x) is defined, abbreviated dom(f)

We will study real valued functions ($\mathrm{range}(f) \subseteq \mathbb{R}$) defined on $\mathbb{R}$ ($\mathrm{dom}(f) \subseteq \mathbb{R}$).

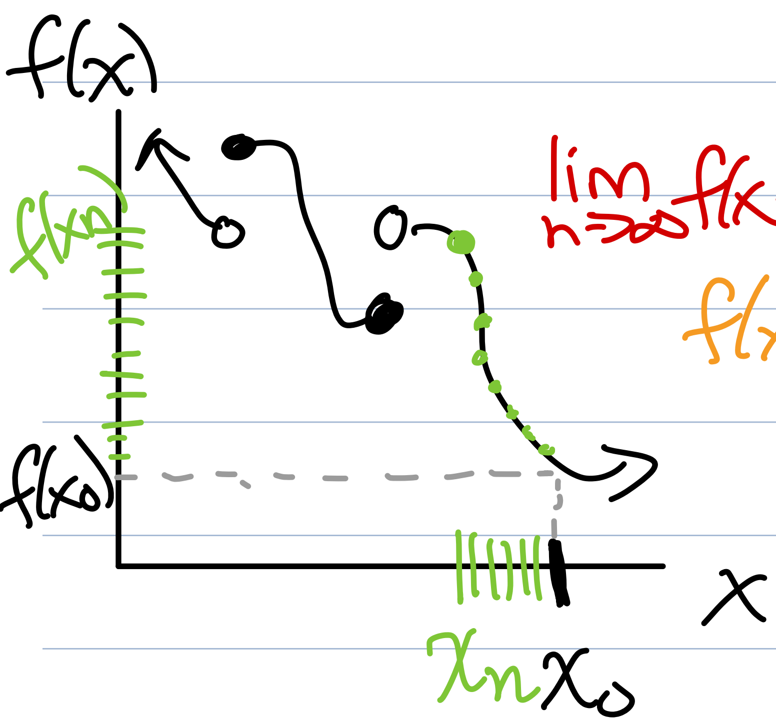
Ex: $f(x) = \frac{1}{x}$, $\mathrm{dom}(f) = \mathbb{R} \setminus \{0\}$

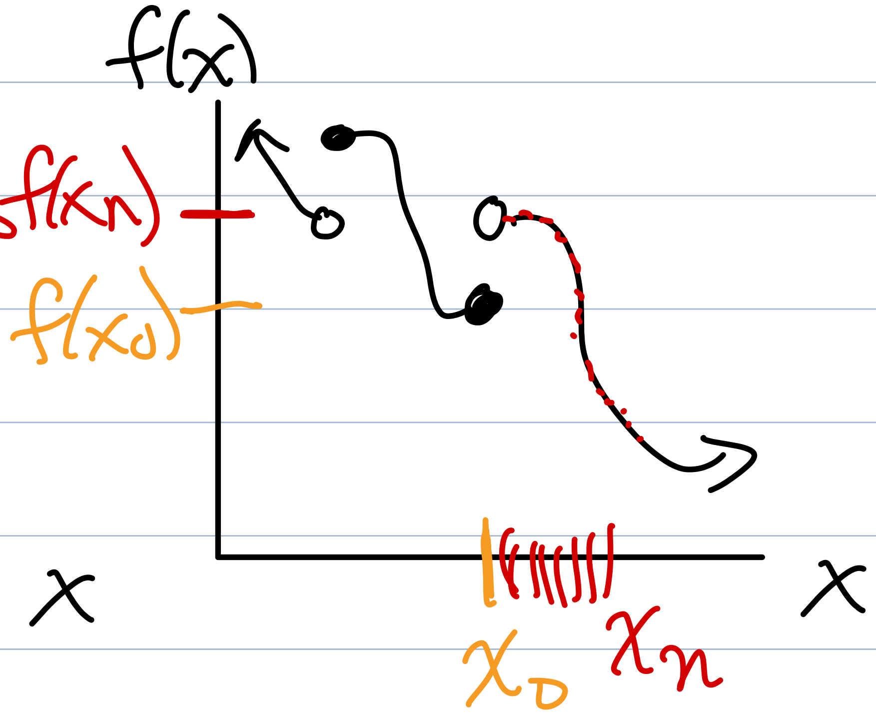
We can make our intuitive notion of continuity precise using sequences:

Def (continuity):

- A function f is continuous at a point $x_0 \in \mathrm{dom}(f)$ if, for every sequence x_n in $\mathrm{dom}(f)$ satisfying $\lim_{n\to\infty} x_n = x_0$, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.

CONTINUOUS AT x_0 DISCTS AT x_0





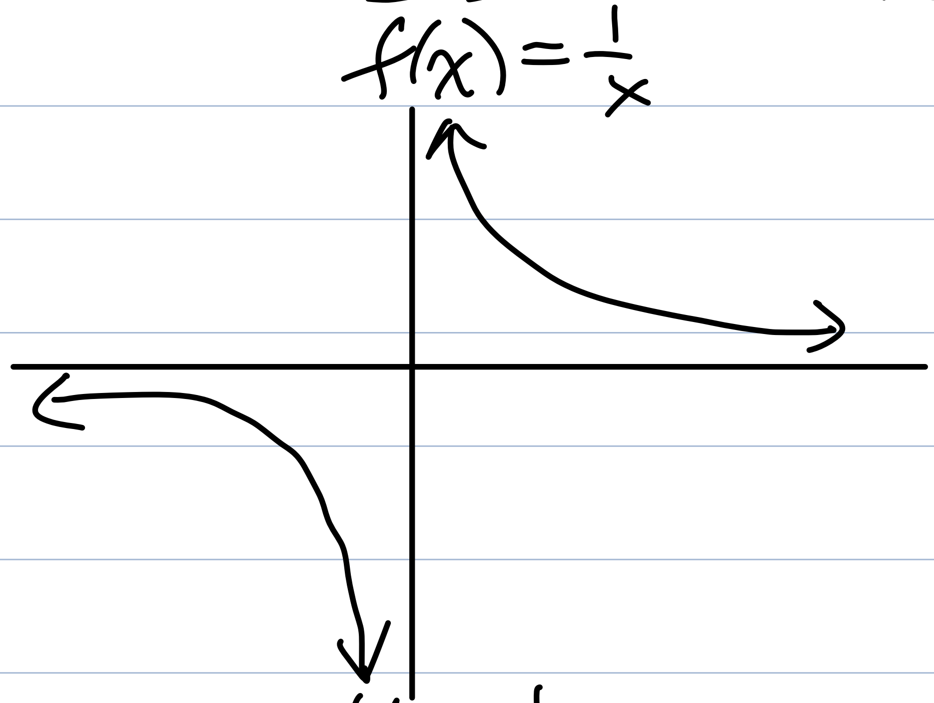
We expect f is cts at x_0

For a function to be discontinuous at x_0, I only need one x_n in $\mathrm{dom}(f)$ s.t. $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} f(x_n) \neq f(x_0)$.

Def (continuity, continued):

- f is <u>continuous on a set $S \subseteq \text{dom}(f)$</u> if it is continuous at every point in S.
- f is <u>continuous</u> if it is continuous on all of $\text{dom}(f)$.

Question: Is $f(x) = \frac{1}{x}$ continuous?



Answer: Yes!

To show this, we must show that f is cts at all

$x_0 \in \text{dom}(f) = \mathbb{R}\setminus\{0\}$.

Fix $x_0 \in \mathbb{R}\setminus\{0\}$ arbitrary. Suppose x_n is a sequence in $\mathbb{R}\setminus\{0\}$ satisfying $\lim_{n\to\infty} x_n = x_0$.

We must show $\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{x_0}$. This follows from the result that the limit of the quotient is the quotient of the limits (as long as the denominator is nonzero).

This is slightly counterintuitive: the break in dom(f) at $x=0$ allows a "jump" at that point.

Remark: A function $f(x)$ is continuous exactly when you can "pass the limit inside the function," that is, if x_n and x_0 are in $\mathrm{dom}(f)$ and $\lim_{n\to\infty} x_n = x_0$, then

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(x_0).$$

The most common definition of cty is the ε-δ definition, which is equivalent to our defn.

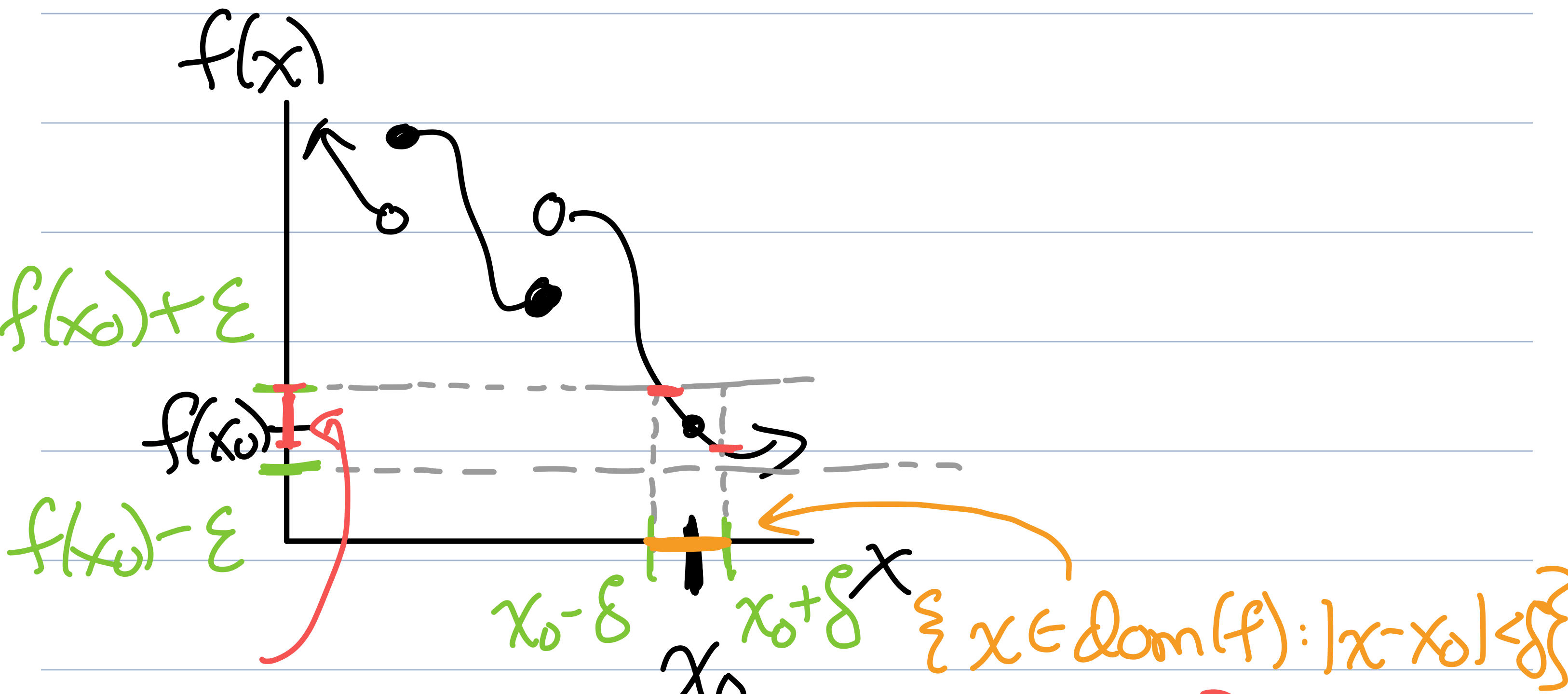
Thm: Given f and $x_0 \in \text{dom}(f)$

(I) $\boxed{f \text{ is continuous at } x_0}$ iff

for all $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in \text{dom}(f)$ and $|x - x_0| < \delta$ imply $|f(x) - f(x_0)| < \varepsilon$.

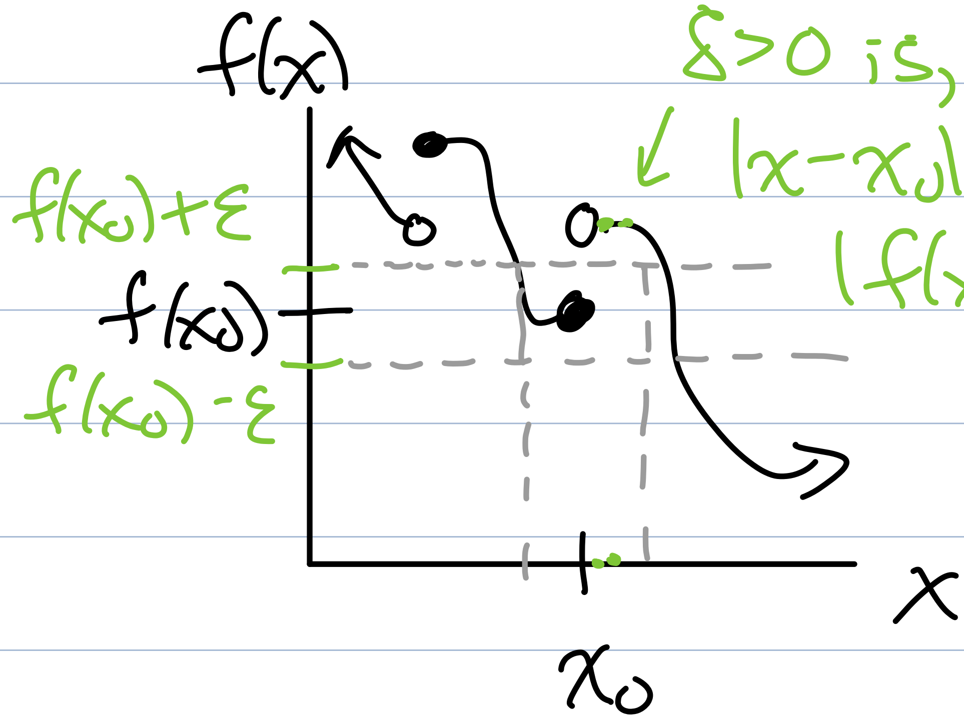
(II)

Mental image



$\{f(x) : x \in \text{dom}(f) \text{ and } |x - x_0| < \delta\}$

No matter how small $\delta > 0$ is, $\exists\, x \in \text{dom}(f)$, $|x - x_0| < \delta$ but $|f(x) - f(x_0)| \geq \varepsilon$.



To show f is discontinuous at x_0, it suffices to find some $\varepsilon > 0$ so that $\forall\, \delta > 0$, $\exists\, x \in \text{dom}(f)$ s.t. $|x - x_0| < \delta$ and $|f(x) - f(x_0)| \geq \varepsilon$.

Pf of Thm: Fix f and $x_0 \in \text{dom}(f)$.

First, assume (II). WTS (I).

Fix x_n in dom(f) s.t. $\lim_{n\to\infty} x_n = x_0$.
We must show $\lim_{n\to\infty} f(x_n) = f(x_0)$.

Fix $\varepsilon > 0$ arbitrary. We must show $\exists$ N s.t. $n > N$ ensures $|f(x_n) - f(x_0)| < \varepsilon$.

By II, $\exists\, \delta > 0$ s.t. $x \in \mathrm{dom}(f)$ and $|x - x_0| < \delta$ imply $|f(x) - f(x_0)| < \varepsilon$.

Since x_n converges to x_0 $\exists$ N s.t. $n > N$ ensures $|x_n - x_0| < \delta \Rightarrow |f(x_n) - f(x_0)| < \varepsilon$.

This shows I holds.

It remains to show that
(I) $\Rightarrow$ (II). We will show
$\neg$(II) $\Rightarrow$ $\neg$(I).

Assume $\neg$(II), that is, assume
$\exists\, \varepsilon > 0$ s.t. $\forall\, \delta > 0$, there
exists $x \in \text{dom}(f)$ s.t. $|x - x_0| < \delta$
and $|f(x) - f(x_0)| \geq \varepsilon$. $(*)$

We must find some sequence
x_n in dom(f) s.t. $\lim\limits_{n\to\infty} x_n = x_0$
s.t. $\lim\limits_{n\to\infty} f(x_n) \neq f(x_0)$.

By $(*)$ with $\delta = \frac{1}{n}$, for all $n \in \mathbb{N}$,
there exists $x_n \in \text{dom}(f)$
s.t. $|x_n - x_0| < \frac{1}{n}$ and $|f(x_n) - f(x_0)| \geq \varepsilon$.

$\Updownarrow$

$x_0 - \frac{1}{n} < x_n < x_0 + \frac{1}{n}$

By Squeeze Lemma, $\lim_{n\to\infty} x_n = x_0$

Since $|f(x_n) - f(x_0)| \geq \varepsilon \ \forall n \in \mathbb{N}$, $f(x_n)$ does not converge to $f(x_0)$.

Thus $\neg$ (I) holds. $\square$

Ex: Consider $f(x) = 3x^2 - 2$, $\mathrm{dom}(f) = \mathbb{R}$

WTS f is continuous.
Show via sequences defn...
Fix $x_0 \in \mathbb{R}$ arbitrary and x_n converging to x_0.

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 3x_n^2 - 2 \overset{\text{limit theorems}}{=} 3x_0^2 - 2 = f(x_0)$$

Next time, we will contrast this with ε-δ defn of cty.