Lecture 13
CS 117, S25
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Recall:

Def (continuity):

- A function f is continuous at a point $x_0 \in \text{dom}(f)$ if, for every sequence x_n in $\text{dom}(f)$ satisfying $\lim_{n\to\infty} x_n = x_0$, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.
- f is continuous on a set $S \subseteq \text{dom}(f)$ if it is continuous at every point in S.
- f is continuous if it is continuous on all of $\text{dom}(f)$.

MAJOR THM #6 (?)

<u>Thm</u>: Given f and $x_0 \in \mathrm{dom}(f)$

I f is continuous at x_0 iff

for all $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in \mathrm{dom}(f)$ and $|x - x_0| < \delta$ imply $|f(x) - f(x_0)| < \varepsilon$.

II

Warmup example:
Show $f(x) = 3x$ is cts
$\mathrm{dom}(f) = \mathbb{R}$.

Fix $x_0 \in \mathbb{R}$. Fix $\varepsilon > 0$ arbitrary.
Let $\delta = \varepsilon/3$. Then

$$
\begin{aligned}
|x - x_0| < \delta &\iff |x - x_0| < \varepsilon/3 \\
&\iff |3x - 3x_0| < \varepsilon \\
&\iff |f(x) - f(x_0)| < \varepsilon
\end{aligned}
$$

Ex: $f(x) = 3x^2 - 2$, $\mathrm{dom}(f) = \mathbb{R}$

Last time, we proved that f is cts via sequences defn of cty.

Now, prove the same thing via ε-δ characterization.

Fix $x_0 \in \mathbb{R}$ and $\varepsilon > 0$ arbitrary.

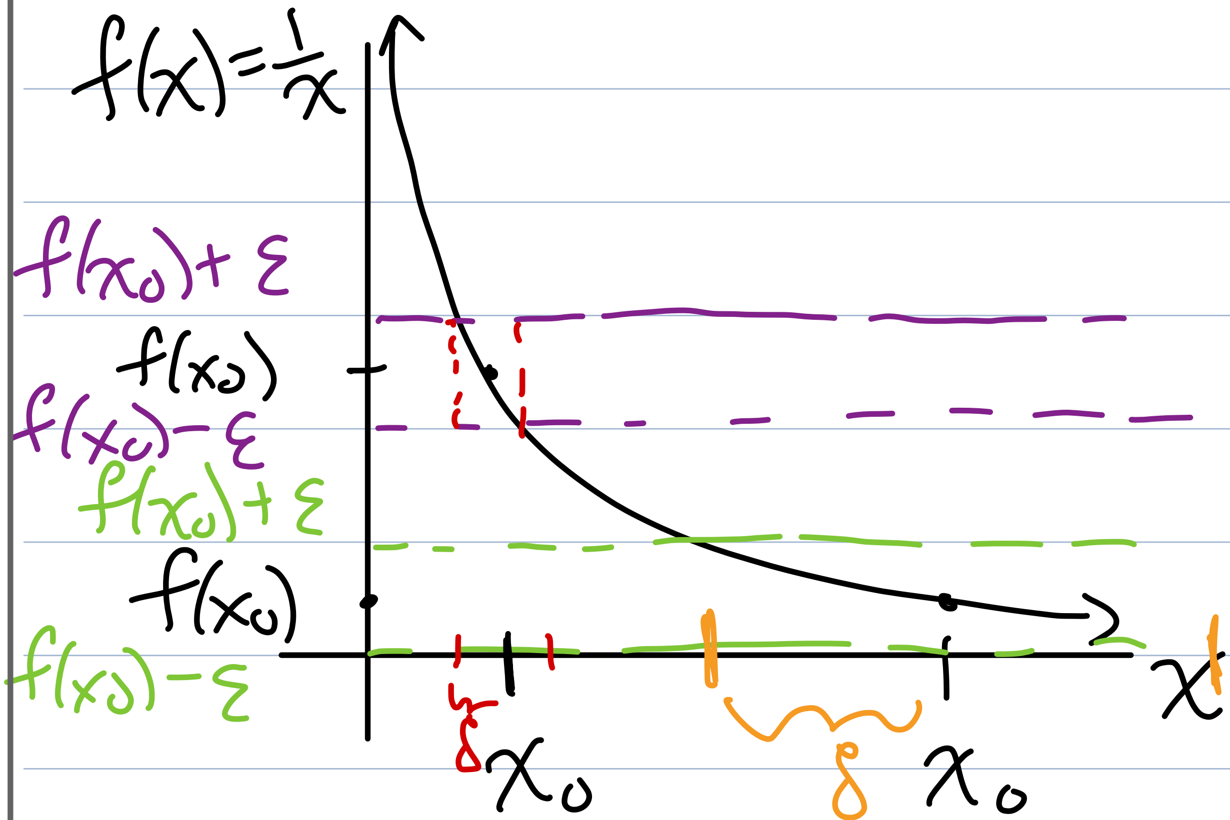
Note that

$$|f(x) - f(x_0)| < \varepsilon$$
$$\iff |(3x^2 - 2) - (3x_0^2 - 2)| < \varepsilon$$
$$\iff 3|x^2 - x_0^2| < \varepsilon$$
$$\iff 3|x - x_0||x + x_0| < \varepsilon$$
$$\impliedby 3|x - x_0|(|x| + |x_0|) < \varepsilon$$
$$\impliedby 3|x - x_0|(2|x_0| + 1) < \varepsilon$$

Triangle ineq: $|x + x_0| \leq |x| + |x_0|$

By $(*)$

Mental image: it's okay if choice of δ depends on x_0



which will be true if the $\delta > 0$ we choose is less than 1

If $|x - x_0| \leq 1$, the reverse triangle inequality ensures

$$||x| - |x_0|| \leq 1 \quad (*)$$

$$\Leftrightarrow -1 \leq |x| - |x_0| \leq 1 \Leftrightarrow |x_0| - 1 \leq |x| \leq |x_0| + 1$$

Thus,

$$|f(x)-f(x_0)|<\varepsilon$$

$$\Leftarrow |x-x_0| < \frac{\varepsilon}{\underbrace{3(2|x_0|+1)}}$$

we want δ to be ≤ 1 and $\leq \frac{\varepsilon}{3(2|x_0|+1)}$

Thus, for $\delta := \min\left\{\frac{\varepsilon}{3(2|x_0|+1)}, 1\right\}$

Then $|x-x_0|<\delta$ implies $|f(x)-f(x_0)|<\varepsilon$. This shows f is cts at x_0. Since x_0 was arbitrary, this shows f is continuous.

In analogy with limit theorems for sequences, we want to show functions are cts by decomposing them into simpler parts that are "obviously" cts.

But first... combining simple fns into more complicated fns.

$(f+g)(x) = f(x) + g(x),$

$\qquad \mathrm{dom}(f+g) = \mathrm{dom}(f) \cap \mathrm{dom}(g)$

$(fg)(x) = f(x)g(x)$

$\qquad \mathrm{dom}(fg) = \mathrm{dom}(f) \cap \mathrm{dom}(g)$

$(f/g)(x) = \frac{f(x)}{g(x)}$,

$\mathrm{dom}(f/g) = \mathrm{dom}(f) \cap \mathrm{dom}(g) \cap \{x : g(x) \neq 0\}$

$(f \circ g)(x) = f(g(x))$

$\mathrm{dom}(f \circ g) = \mathrm{dom}(g) \cap \{x : g(x) \in \mathrm{dom}(f)\}$

<u>Thm</u>: If f and g are continuous at $x_0 \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ then

(a) $f+g$ is continuous at x_0

(b) fg is continuous at x_0

(c) f/g is continuous at x_0, as long as $g(x_0) \neq 0$.

<u>Pf</u>: Let $h_1 = f+g$, $h_2 = fg$, $h_3 = f/g$. Fix x_0 as in theorem, that is $x_0 \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ for $i = 1, 2,$

and $g(x_0) \neq 0$ for $i=3$.
Let x_n be a sequence in dom(h_i) that converges to x_0. We must show $h_i(x_n)$ converges to $h_i(x_0)$.

Since f and g are cts,
$\lim_{n \to \infty} f(x_n) = f(x_0)$, $\lim_{n \to \infty} g(x_n) = g(x_0)$

For $i=1$, $\lim_{n \to \infty} h_1(x_n) = \lim_{n \to \infty} f(x_n) + g(x_n)$
$= f(x_0) + g(x_0) = h_1(x_0)$,
since the limit of sum is sum of limits.

For $i=2$, same argument, since limit of product is product of limits.

For $i=3$, $\lim_{n\to\infty} h_3(x_n)$

$$= \lim_{n\to\infty} f(x_n)/g(x_n)$$

$$= \frac{\lim_{n\to\infty} f(x_n)}{\lim_{n\to\infty} g(x_n)}$$

$$= f(x_0)/g(x_0)$$

$$= h_3(x_0),$$

since the limit of the quotient is the quotient of the limits. $\square$

Thm: Suppose g is continuous at x_0 and f is continuous at $g(x_0)$. Then $f \circ g$ is continuous at x_0.

Pf: Suppose $x_n \in \mathrm{dom}(f \circ g)$ converges to x_0. Since g is cts at x_0, $\lim_{n\to\infty} g(x_n) = g(x_0)$. Since f is continuous at $g(x_0)$, $\lim_{n\to\infty} f(g(x_n)) = f(g(x_0))$. $\square$

Next key property of functions: boundedness.

Def: f is <u>bounded on $S \subseteq \mathrm{dom}(f)$</u> if there exists $M > 0$ s.t. $|f(x)| \leq M$ for all $x \in S$.
We say f is <u>bounded</u> if f is bounded on $\mathrm{dom}(f)$.

Remark:

- s_n is a bounded sequence

$\Updownarrow$

$\{s_n : n \in \mathbb{N}\}$ is a bounded set

- f is a bounded function fn

$\Updownarrow$

$\{f(x) : x \in dom(f)\}$ is a bounded set

$\quad\quad = image(f)$

Thm: A continuous function f on a closed interval $[a,b] \subseteq dom(f)$ attains its maximum and minimum.

That is to say... "the maximum of f on $[a,b]$"

- $\max\{f(x) : x \in [a,b]\}$ and "the minimum"...
$\min\{f(x) : x \in [a,b]\}$ exist

"a maximizer of f on [a,b]"

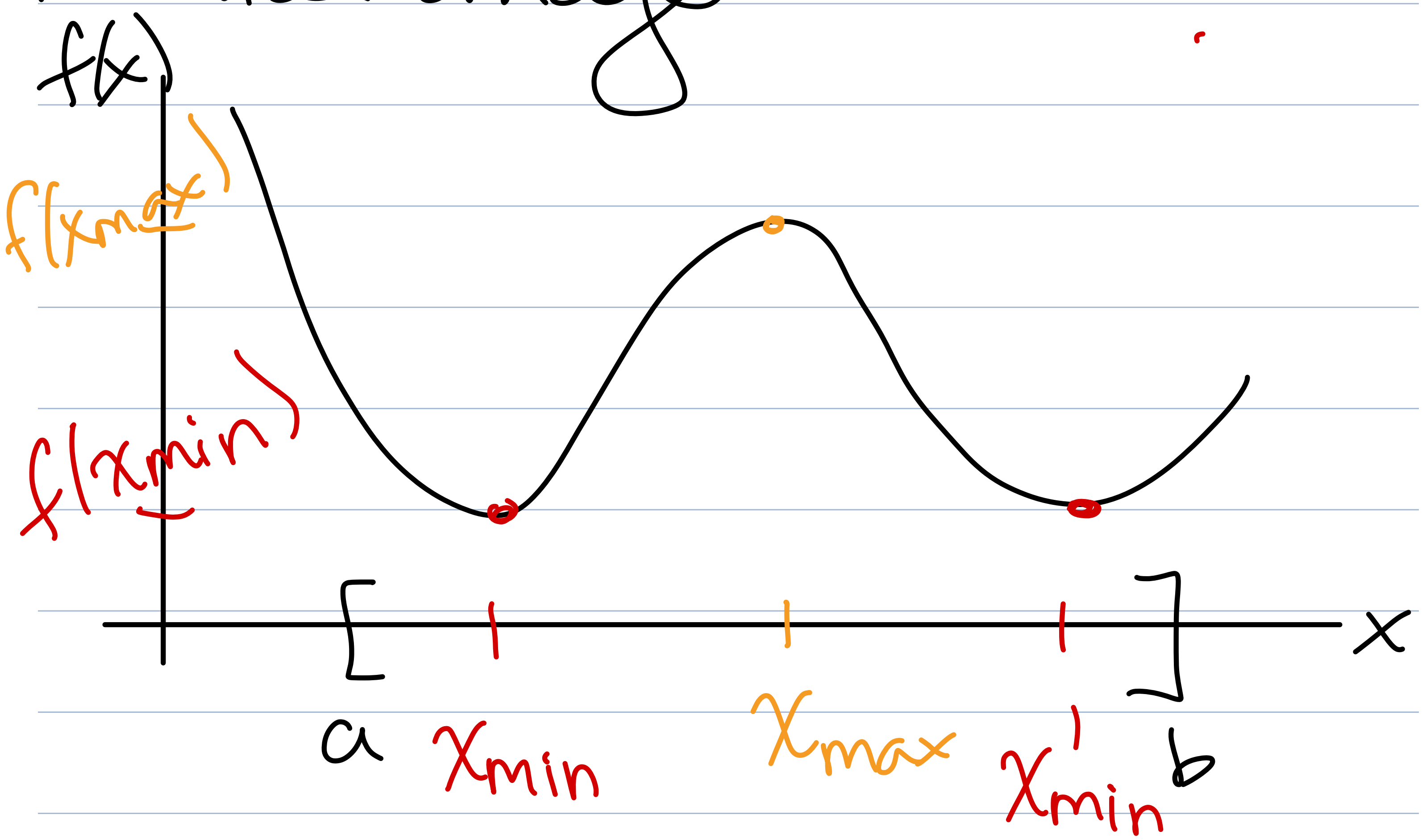
"a minimizer of f on [a,b]"

- $\exists\ x_{max}, x_{min} \in [a,b]$ so that

$$f(x_{max}) = \max\{f(x) : x \in [a,b]\}$$

$$f(x_{min}) = \min\{f(x) : x \in [a,b]\}$$

Mental image:



Rmk: An immediate consequence of this theorem is that any cts fn f is bounded on any closed interval $[a,b] \subseteq \mathrm{dom}(f)$, since

$$f(x_{min}) \le f(x) \le f(x_{max})$$

$\forall x \in [a,b]$.

What goes wrong if the interval is not closed?