Lecture 14

CS 117, S25

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Announcement: Midterm 2 is one week from Thurs, no HW: practice midterm this week

Recall:

Thm: If f and g are continuous at $x_0 \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ then

(a) $f+g$ is continuous at x_0

(b) fg is continuous at x_0

(c) f/g is continuous at x_0, as long as $g(x_0) \neq 0$.

Thm: Suppose g is continuous at x_0 and f is continuous at $g(x_0)$. Then $f \circ g$ is continuous at x_0.

Def: f is bounded on $S \subseteq \mathrm{dom}(f)$ if there exists $M>0$ s.t. $|f(x)| \le M$ for all $x \in S$.
We say f is bounded if f is bounded on $\mathrm{dom}(f)$.

Thm: A continuous function f on a closed interval $[a,b] \subseteq \mathrm{dom}(f)$ attains its maximum and minimum.

That is to say... "the maximum of f on $[a,b]$"

- $\max\{f(x) : x \in [a,b]\}$ and $\min\{f(x) : x \in [a,b]\}$ exist "the minimum ..."

"a maximizer of f on [a,b]"

"a minimizer of f on [a,b]"

- $\exists\ x_{max}, x_{min} \in [a,b]$ so that

$$f(x_{max}) = \max\{f(x) : x \in [a,b]\}$$

$$f(x_{min}) = \min\{f(x) : x \in [a,b]\}$$

Pf: Last time, we show f is bounded on [a,b]

Now, we show that f attains its maximum, via considering a <u>maximizing sequence</u>.

Since $\{f(x) : x \in [a,b]\}$ is a bounded subset of real numbers, its supremum exists.

Let $M := \sup \underbrace{\{f(x) : x \in [a,b]\}}_{S}$.

There exists $f(x_n)$ for $x_n \in [a,b]$ so that

a sequence of points in S

(*) $\lim_{n \to \infty} f(x_n) = M.$ ← that converge to sup(S)

Since x_n is a bounded sequence, by Bolzano-Weierstrass it has a convergent subsequence x_{n_k}, with

$$\lim_{k \to \infty} x_{n_k} = x_0,$$

for $x_0 \in [a,b]$. Since f is cts, $\lim_{k \to \infty} f(x_{n_k}) = f(x_0)$. Also, by (*), we have $\lim_{k \to \infty} f(x_{n_k}) = M$

Therefore,

$$f(x_0) = M = \sup(\{f(x) : x \in [a,b]\})$$

Thus $f(x_0) = \max\{f(x) : x \in [a,b]\}$.

It remains to show that f attains it minimum on $[a,b]$.

Since f is cts, so is $-f$, and $\mathrm{dom}(-f) = \mathrm{dom}(f) \supseteq [a,b]$.
We have just shown that any cts fn attains its max on a closed interval in its domain, so $\exists\ x_1 \in [a,b]$ s.t.

$$-f(x_1) = \max\{-f(x) : x \in [a,b]\}$$

$$= -\min\{f(x) : x \in [a,b]\}$$

Thus, f attains its minimum at x_1. $\square$

An immediate consequence of this is a familiar result...

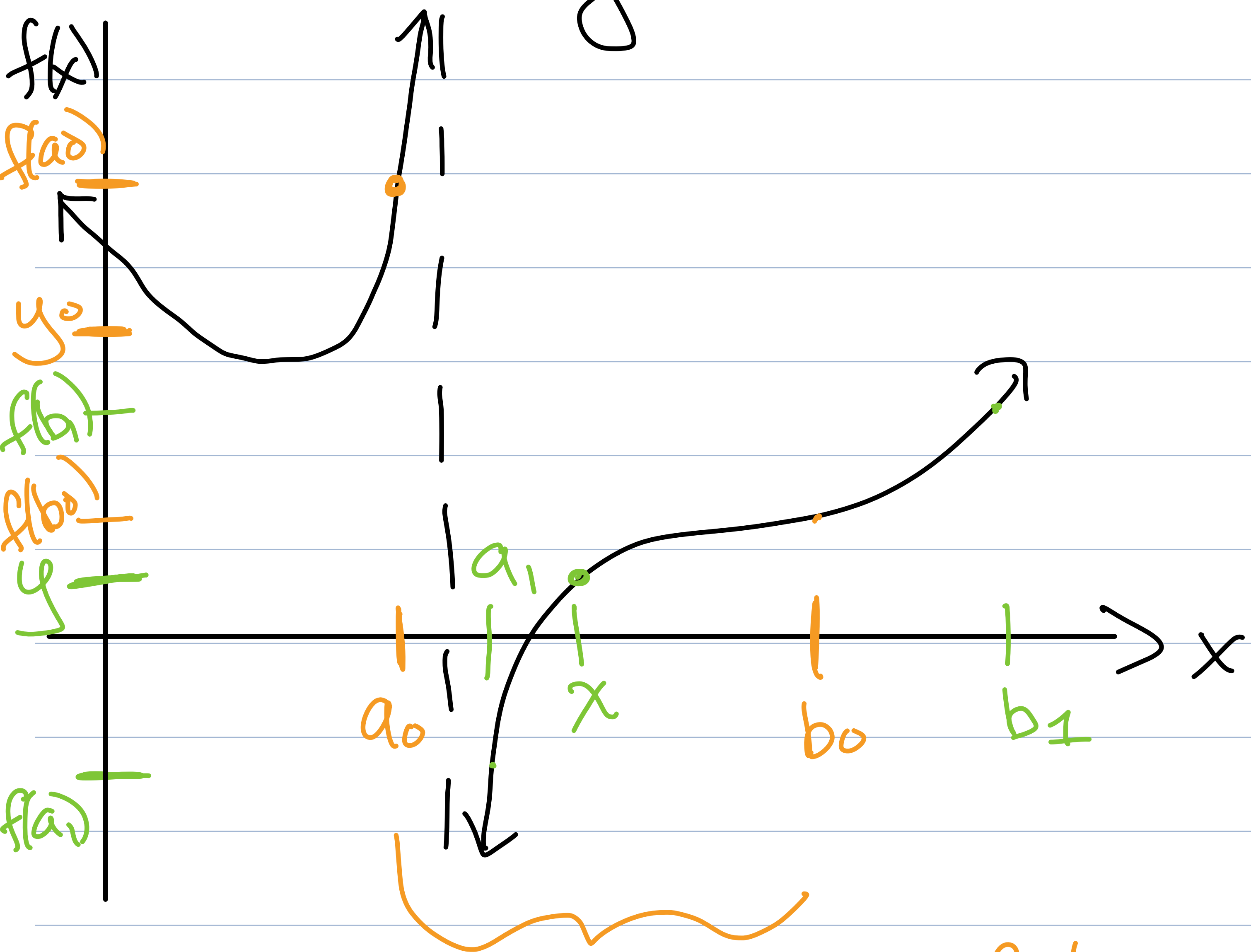
MAJOR THM 8

Thm (Intermediate Value Thm)
If f is continuous on an interval $I \subseteq \text{dom}(f)$, then for all $a, b \in I$, if y lies between $f(a)$ and $f(b)$, then there exists x between a and b s.t. $f(x) = y$.

either
$f(a) \leq y \leq f(b)$
or $f(b) \leq y \leq f(a)$

either
$a \leq x \leq b$
or $b \leq x \leq a$

Mental image:

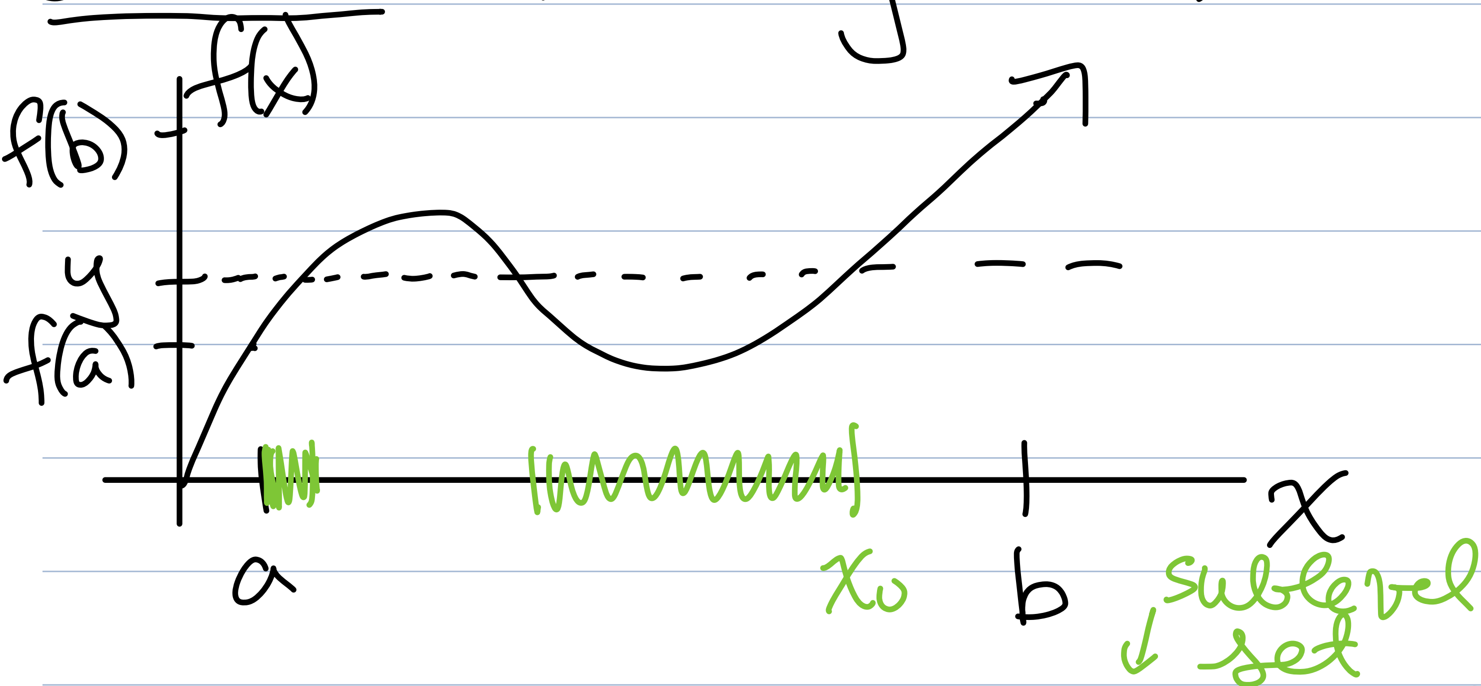


no x between a_0 and b_0
s.t. $f(x) = y_0$

"continuous fns can't have jumps"

Pf: Fix $a, b \in I \subseteq \mathrm{dom}(f)$, where f is continuous on I.
Assume WLOG $a \leq b$.
Suppose y lies between $f(a)$ and $f(b)$. WTS $\exists\, x \in [a,b]$ s.t. $f(x) = y$.

Case 1: $f(a) \leq y \leq f(b)$



Define $S = \{x \in [a,b] : f(x) \leq y\}$
Since S is bounded, its supremum exists. Let $x_0 := \sup(S)$.
Note that $x_0 \in [a,b]$.

Choose x_n s.t. $x_n \in S \ \forall n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x_0$.

Since f is cts, $\lim_{n\to\infty} f(x_n) = f(x_0)$

WTS $f(x_0) = y$.

Since $x_n \in S \ \forall n \in \mathbb{N}$, $f(x_n) \leq y$ $\forall n \in \mathbb{N}$. Thus $f(x_0) \leq y$.

It remains to show $f(x_0) \geq y$. If $x_0 = b$, by our hypothesis in Case 1, $y \leq f(b) = f(x_0)$, and we are done. Thus, we may assume $x_0 < b$.

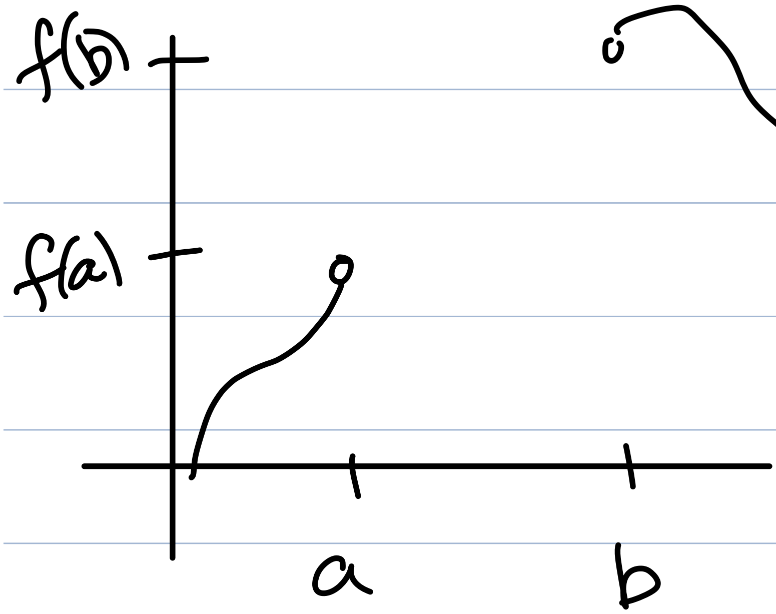
Define $t_n = \min\{x_0 + \frac{1}{n}, b\}$. By defn, $\lim_{n\to\infty} t_n = x_0$.

Since $t_n > x_0 = \sup(S)$, $t_n \notin S$, and $t_n \in [a,b]$, so $f(t_n) > y$ for all $n \in \mathbb{N}$.

Since f is cts, $\lim_{n\to\infty} f(t_n) = f(x_0)$, so $f(x_0) \geq y$. This completes the proof of Case 1.

<u>Case 2</u>: $f(b) \leq y \leq f(a)$

Then $-f(a) \leq -y \leq -f(b)$. By Case 1 applied to the cts fn $-f$, $\exists\ x_0 \in [a,b]$ s.t. $-f(x_0) = -y$, so $f(x_0) = y$. $\square$



Uniform continuity

Recall.

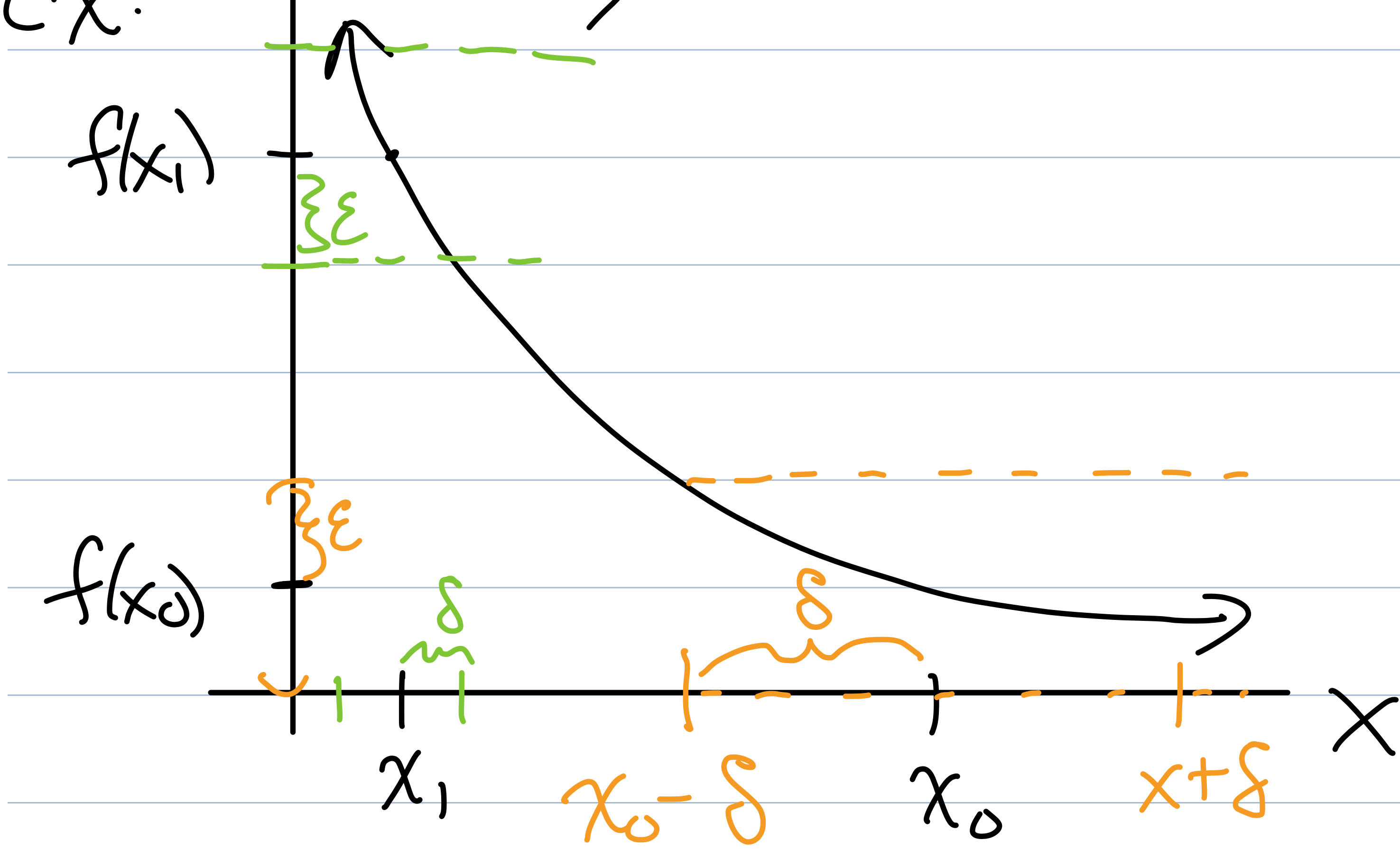
Thm (ε-δ characterization of cty):
Given f and $x_0 \in \mathrm{dom}(f)$

f is cts at x_0 if and only if

for all $\varepsilon > 0$, there exists $\delta > 0$ s.t.
$x \in \mathrm{dom}(f)$ and $|x - x_0| < \delta$
imply $|f(x) - f(x_0)| < \varepsilon$.

In general, the choice of δ depends on both ε and x_0.

Ex: $f(x) = \frac{1}{x}$



Functions are uniformly cts when you can find a $\delta > 0$ that works for all x_0.

Def: A function f is <u>uniformly continuous</u> on $S \subseteq \mathrm{dom}(f)$ if
$\forall \varepsilon > 0$, $\exists\ \delta > 0$ s.t.
$x, y \in S$ and $|x-y| < \delta$
imply $|f(x) - f(y)| < \varepsilon$

Rmk: if f is uniformly continuous on $S \subseteq \mathrm{dom}(f)$, then f is cts at all $x_0 \in S$.

We have proved that $f(x) = \frac{1}{x}$ is continuous on $\mathrm{dom}(f) = \mathbb{R} \setminus \{0\}$.

We will now prove it is <u>not</u> uniformly continuous on $\mathrm{dom}(f) = \mathbb{R} \setminus \{0\}$.

Assume, for the sake of contradiction, that f is uniformly continuous on $\mathrm{dom}(f) = \mathbb{R} \setminus \{0\}$.

Let $\varepsilon = 1$. By defn, $\exists\, \delta > 0$ s.t. $x, y \in \mathbb{R} \setminus \{0\}$ and $|x-y| < \delta$ imply

$$|f(x) - f(y)| < 1.$$
$$\Updownarrow$$
$$\left|\frac{1}{x} - \frac{1}{y}\right| < 1$$

Suppose $x = \frac{1}{n}$, $y = \frac{1}{n+1}$. Then $|x-y| < \frac{1}{n}$ and $\left|\frac{1}{x} - \frac{1}{y}\right| = |n - (n+1)| = 1$. Thus, choosing n sufficiently large so $\frac{1}{n} < \delta$ gives a contradiction.