Lecture 15

CS 117, S25

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Announcement:

- Midterm 2 one week from today
- No HW this week - practice midterm 2

Recall:

MAJOR THM 8

Thm (Intermediate Value Thm)

If f is continuous on an interval $I \subseteq \text{dom}(f)$, then for all $a, b \in I$, if y lies between $f(a)$ and $f(b)$, then there exists x between a and b s.t. $f(x) = y$.

END OF MATERIAL

FOR MIDTERM 2

Def: A function f is <u>uniformly continuous</u> on $S \subseteq \text{dom}(f)$ if $\forall \varepsilon > 0$, $\exists\, \delta > 0$ s.t.

$x, y \in S$ and $|x-y| < \delta$
imply $|f(x) - f(y)| < \varepsilon$

Last time, we proved that $f(x) = \frac{1}{x}$ is <u>not</u> uniformly continuous on $\text{dom}(f) = \mathbb{R} \setminus \{0\}$.

On the other hand, f is uniformly continuous on $[a,b]$, for any $[a,b] \subseteq \text{dom}(f)$.

To see this, fix $\varepsilon > 0$ arbitrary.

For any $z \in [a,b]$,

$$|z| \geq \min\{|a|, |b|\}.$$

Let $\delta := \varepsilon (\min\{|a|,|b|\})^2$. Then...

(Read from bottom to top)

$$|f(x) - f(y)| < \varepsilon$$

$$\Updownarrow$$

$$\left|\frac{1}{x} - \frac{1}{y}\right| < \varepsilon$$

$$\Updownarrow$$

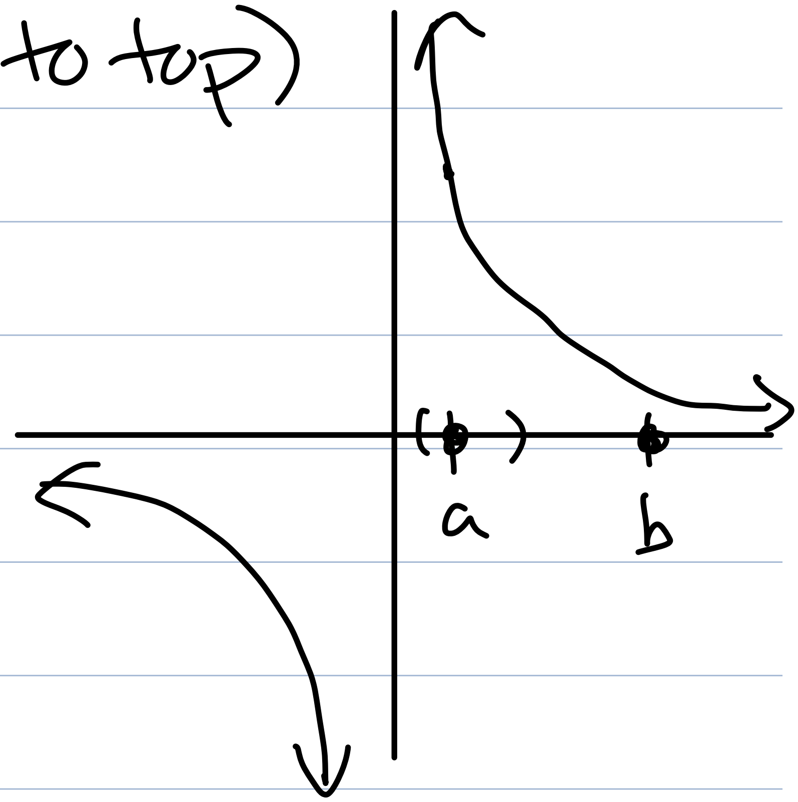
$$\left|\frac{y-x}{xy}\right| < \varepsilon$$

$$\Updownarrow$$

$$|y-x| < \varepsilon |x||y|$$

$$\Uparrow$$

$$|y-x| < \varepsilon (\min\{|a|,|b|\})^2$$



Therefore, f is unif continuous on $[a,b]$.

In fact, the analogue of this holds for any ~~cts fn~~ cts fn f.

MAJOR THM 9

Thm: If f is cts on a closed interval $[a,b] \subseteq \mathrm{dom}(f)$, then f is unif cts on $[a,b]$.

A quick review of earlier material...

- If $s_n \in [a,b]$ $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n = s$, then $s \in [a,b]$.

$r_n \leq t_n$ $\forall n \in \mathbb{N}$ $\Rightarrow \lim_{n\to\infty} r_n \leq \lim_{n\to\infty} t_n$

limits exist

If $s_n \in (a,b)$ $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n = s$, then $s \in (a,b)$.

False! $\frac{1}{n} \in (0,2)$, but $\lim_{n\to\infty} \frac{1}{n} \notin (0,2)$

Pf of Thm:

Suppose f is cts on a closed interval $[a,b] \subseteq \mathrm{dom}(f)$.

Assume, for the sake of contradiction, that f is not unif cts on $[a,b]$, that is, $\exists\, \varepsilon > 0$ s.t. $\forall \delta > 0$, $\exists$ $x, y \in [a,b]$ s.t. $|x-y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$.

In particular, for any $n \in \mathbb{N}$, $\exists\, x_n, y_n \in [a,b]$ s.t. $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \varepsilon$.

Since any bounded sequence has a convergent subsequence, there exist subsequences x_{n_k} and y_{n_k} that converge to limits x_0 & y_0.

Sidenote: Suppose t_n and r_n are bounded sequences. By B.W. $\exists$ subsequence t_{n_k} that converges to t_0. Note that r_{n_k} is still a bdd sequence, so $\exists$ further subseq. $r_{n_{k_\ell}}$ that coverges to r_0. Finally, $t_{n_{k_\ell}}$ converges to t_0.

By Squeeze Lemma

$$y_n - \frac{1}{n} \le x_n \le y_n + \frac{1}{n}$$

ensures $y_0 \le x_0 \le y_0 \Rightarrow x_0 = y_0$.

Furthermore, $x_n \in [a,b]$ $\forall n$ implies $x_0 \in [a,b]$.

Since f is cts on $[a,b]$,

$$\lim_{k\to\infty} f(x_{n_k}) = f(x_0)$$

$$\lim_{k\to\infty} f(y_{n_k}) = f(x_0)$$

Thus, $\lim_{k\to\infty} f(x_{n_k}) - f(y_{n_k}) = 0$,

so $\lim_{k\to\infty} |f(x_{n_k}) - f(y_{n_k})| = 0$.

This contradicts that

$|f(x_{n_k}) - f(y_{n_k})| \ge \varepsilon > 0 \ \forall k \in \mathbb{N}$.

$\square$

Q: If f is cts on $(a,b) \subseteq \mathrm{dom}(f)$, is f unif cts on (a,b)?

A: No, consider $f(x) = \frac{1}{x}$, $\underline{\underline{(a,b) = }}(0,1)$.

Why are unif cts fns important?

Think back to cts fns...

"Continuous functions send convergent sequences with limits in dom(f) to convergent sequences"

That is, if $\lim_{n\to\infty} x_n = x_0$, then $\lim_{n\to\infty} f(x_n) = f(x_0)$

Wait....

$f(x) = \frac{1}{x}$, $x_n = \frac{1}{n}$ ← nice convergent sequence

$f(x_n) = n$ ← does not converge

Interestingly, we don't need qualification $(\ast)$ for unif cts fns.

Thm: If f is a uniformly cts fn on $S \subseteq \mathrm{dom}(f)$ and s_n is a convergent sequence satisfying $s_n \in S$ $\forall n \in \mathbb{N}$, then $f(s_n)$ is convergent.

Ex: A unif cts fn f, a cnvgt sequence s_n, $\lim_{n\to\infty} s_n \notin \text{dom}(f)$.

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ \text{undefined} & \text{if } x = 0 \end{cases}$$

$s_n = \frac{1}{n}$, $0 \notin \text{dom}(f)$.

Pf of Thm: It suffices to prove that $f(s_n)$ is a Cauchy sequence, that is, $\forall\, \varepsilon > 0$ $\exists\, N$ s.t. $n, m > N$ ensures $|f(s_n) - f(s_m)| < \varepsilon$.

Fix $\varepsilon > 0$ arbitrary. Since f is unif cts on S, $\exists\, \delta > 0$, s.t. $x, y \in S$, $|x - y| < \delta$, ensures $|f(x) - f(y)| < \varepsilon$.

Since s_n is convergent, hence Cauchy, $\exists N$ s.t. $n, m > N$ ensures $|s_n - s_m| < \delta$, so $|f(s_n) - f(s_m)| < \varepsilon$. $\square$

What follows cty? Differentiability!

In Calculus,

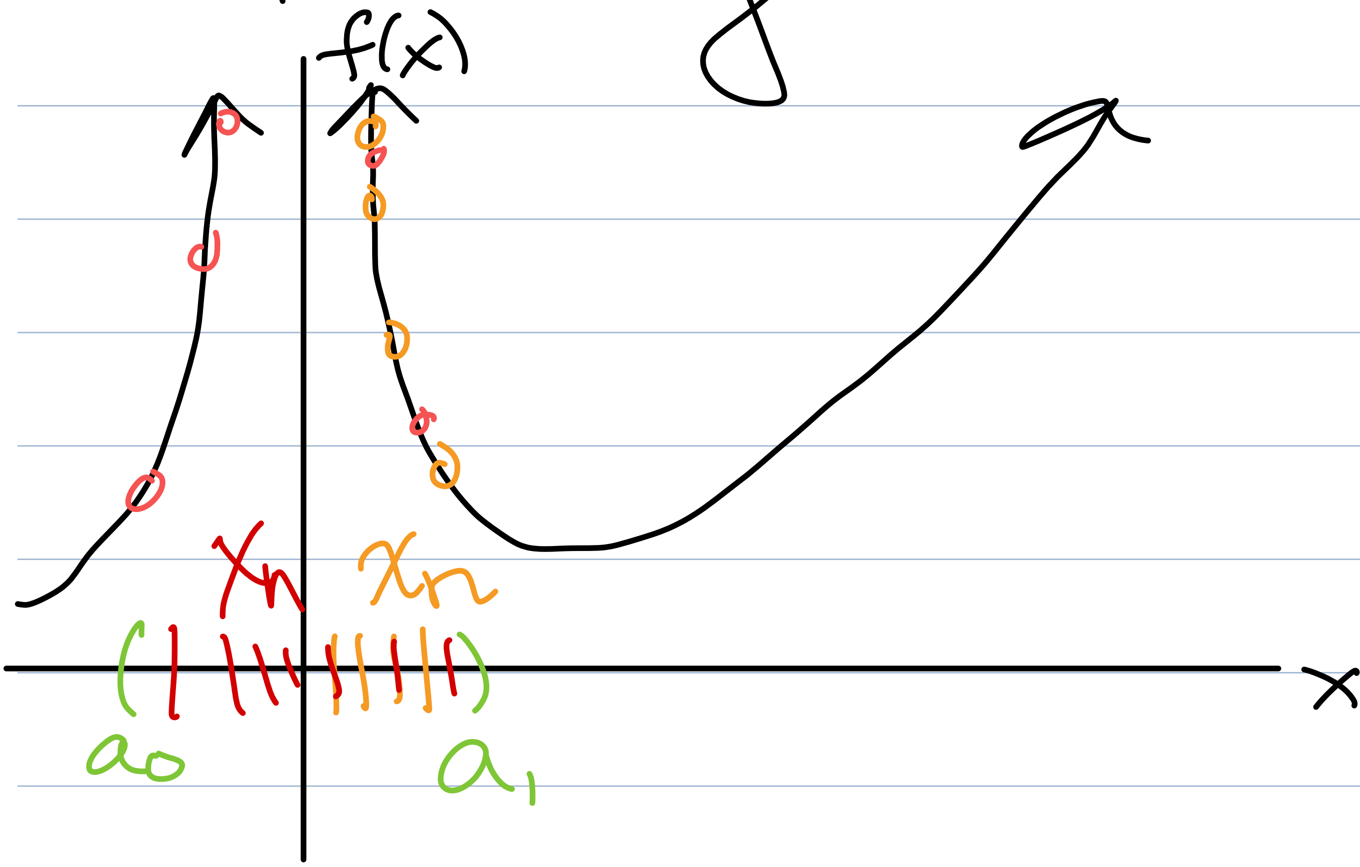
$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}$$

To make sense of this, we must define what it means to take the <u>limit of a function</u>.

Def ("two-sided limit of f at a")
Given a function f and $a \in \mathbb{R}$, we say
$$\lim_{x \to a} f(x) = L$$
If
(i) there exists an interval (a_0, a_1) s.t. $a \in (a_0, a_1)$, $(a_0, a_1) \setminus \{a\} \subseteq \mathrm{dom}(f)$
(ii) for any sequence $x_n \in (a_0, a_1) \setminus \{a\}$ satisfying $\lim_{n \to \infty} x_n = a$, we have $\lim_{n \to \infty} f(x_n) = L$.

Rmk: We don't require $a \in \mathrm{dom}(f)$.
$L \in \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$.

Mental image:

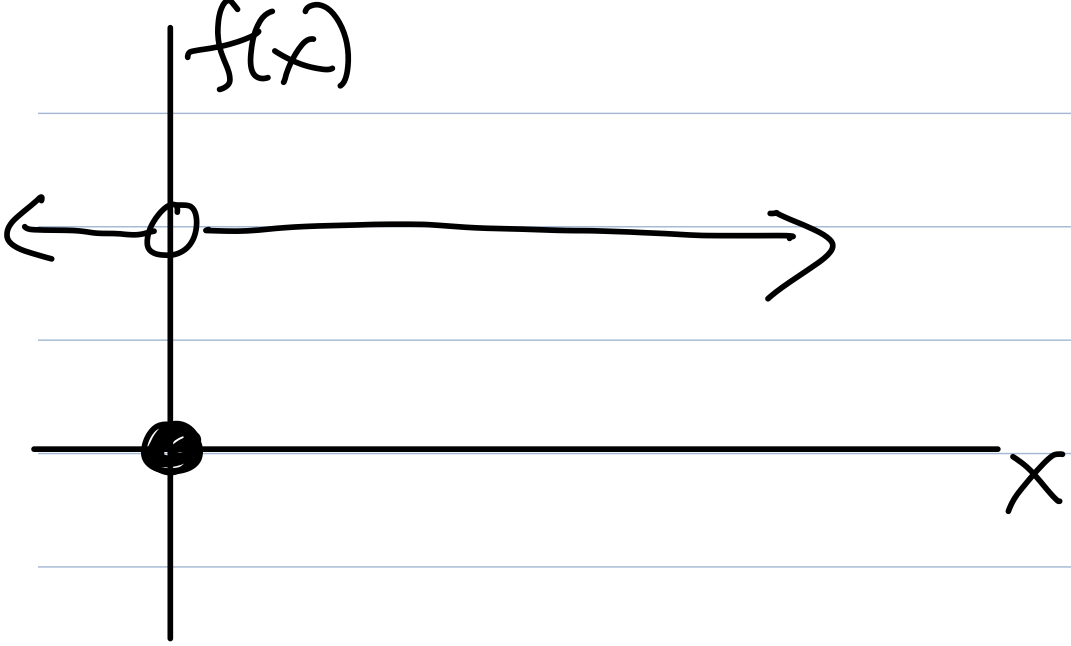


$$\lim_{x \to 0} f(x) = +\infty$$

item (ii) fails, since $\lim_{n\to\infty} f(x_n)$ is different depending on choice of x_n

$\Downarrow$

$$\lim_{x \to 0} f(x) \text{ D.N.E.}$$



$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$\lim_{x \to 0} f(x) = 1 \neq f(0)$$

<u>Rmk</u>: If $a \in (a_0, a_1) \subseteq \text{dom}(f)$ and f is cts on (a_0, a_1), then $\lim_{x \to a} f(x) = f(a)$.

Ex: f is continuous on $\mathrm{dom}(f)$, $a \in \mathrm{dom}(f)$, but $\lim_{x \to a} f(x) \neq f(a)$.

$$f(x) = \begin{cases} 1 & x \in \{0\} \cup [1, +\infty) \\ \text{undefined} & \text{otherwise} \end{cases}$$

$\lim_{x \to 0} f(x)$ D.N.E., since (i) fails.

Idea: no way to approach $a = 0$ via a ~~se~~ sequence in $\mathrm{dom}(f)$.