Lecture 16

CS 117, S25

Announcement:

- Midterm 2 Thursday
- Extra office hours on Wednesday, 2-3pm

Recall:

MAJOR THM 9

Thm: If f is cts on a closed interval $[a,b] \subseteq dom(f)$, then f is unif cts on $[a,b]$.

Thm: If f is a uniformly cts fn on $S \subseteq dom(f)$ and s_n is a convergent sequence satisfying $s_n \in S \ \forall n \in \mathbb{N}$, then $f(s_n)$ is convergent.

Def ("two-sided limit of f at a")
Given a function f and $a \in \mathbb{R}$, we say

$$\lim_{x \to a} f(x) = L$$

"can approach a from LHS and RHS by points in domain of f"

If

(i) there exists an interval (a_0, a_1) s.t. $a \in (a_0, a_1)$, $(a_0, a_1) \setminus \{a\} \subseteq \text{dom}(f)$

(ii) for any sequence $x_n \in (a_0, a_1) \setminus \{a\}$ satisfying $\lim_{n \to \infty} x_n = a$, we have

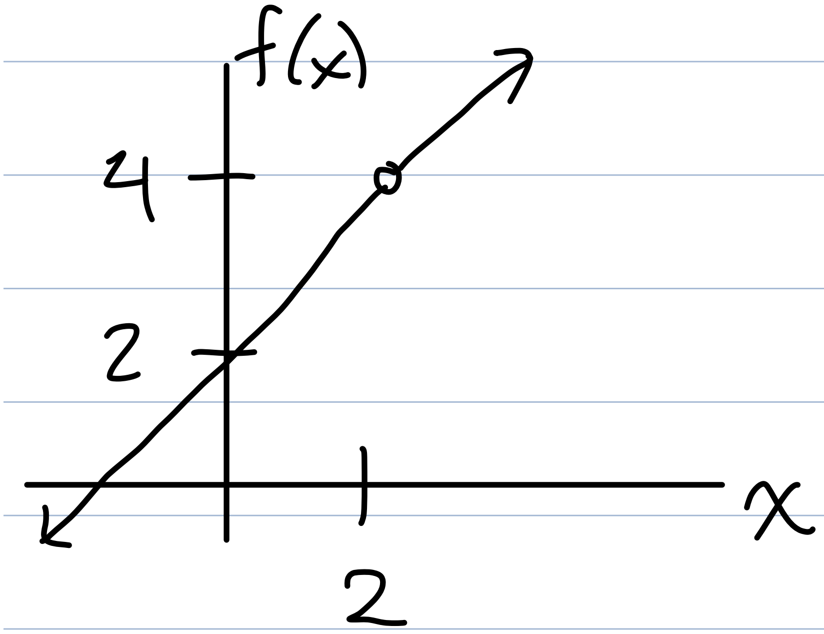
$$\lim_{n \to \infty} f(x_n) = L.$$

"the limiting behavior of $f(x_n)$ is independent of the choice of $x_n \to a$"

Ex: Consider $f(x) = \dfrac{x^2-4}{x-2}$

$$= \frac{(x-2)(x+2)}{x-2}$$

$\mathrm{dom}(f) = \mathbb{R} \setminus \{2\}$



$$\lim_{x \to 2} f(x) = 4$$

Justification:

Let $(a_0, a_1) = (0, 3)$. Then $(0,3) \setminus \{2\} \subseteq \mathrm{dom}(f)$. Let $L = 4$.

Note that, for any $x_n \in (0,3) \setminus \{2\}$ satisfying $x_n \to 2$, we have $f(x_n) = (x_n + 2) \to 4$.

Ex: Suppose f is cts on $(a_0, a_1) \subseteq \text{dom}(f)$ and $a \in (a_0, a_1)$. What is $\lim_{x \to a} f(x)$?

Claim: $\lim_{x \to a} f(x) = f(a)$.

Justification: Part (i) of defn holds by assumption. Since f is cts at a, for any $x_n \in \text{dom}(f)$ s.t. $x_n \to a$, $f(x_n) \to f(a)$. This shows (ii).

Def: Given a function f, $a \in \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$

we say that

Case 1: $a \in \mathbb{R}$

$$\lim_{x \to a^+} f(x) = L$$

Case 2: $a = -\infty$

$$\lim_{x \to -\infty} f(x) = L$$

"x approaches a from above"

Case 1: $a \in \mathbb{R}$

$$\lim_{x \to a^-} f(x) = L$$

Case 2: $a = +\infty$

$$\lim_{x \to +\infty} f(x) = L$$

"x approaches a from below"

if there exists a nonempty open interval

$I = (a, a_1)$

$a_1 \in \mathbb{R}$

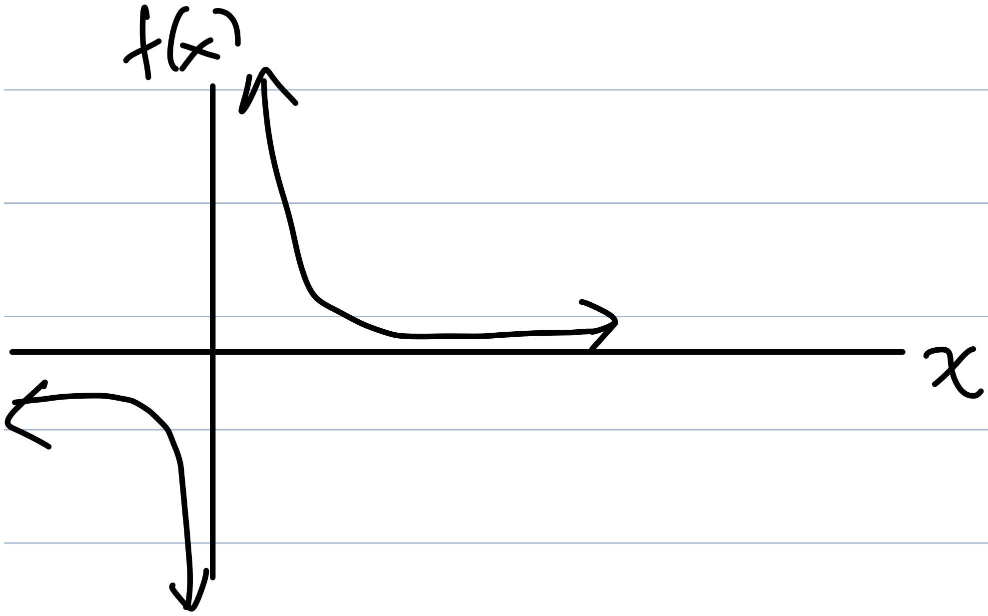
$I = (a_0, a)$

$a_0 \in \mathbb{R}$

s.t. $I \subseteq \mathrm{dom}(f)$ and for every sequence $x_n \in I$ s.t. $\lim_{n \to \infty} x_n = a$, we have $\lim_{n \to \infty} f(x_n) = L$.

Ex: Let $f(x)=\frac{1}{x}$.

$$\lim_{x\to 0^+} f(x)=+\infty$$



Justification: For $I=(0,5)$, we have $I\subseteq \mathrm{dom}(f)$ and for every sequence $x_n\in(0,5)$ s.t. $\lim_{n\to\infty} x_n=0$, we have
$$\lim_{n\to\infty} f(x_n)=\lim_{n\to\infty}\frac{1}{x_n}=+\infty.$$

$$\lim_{x\to 0^-} f(x)=-\infty$$

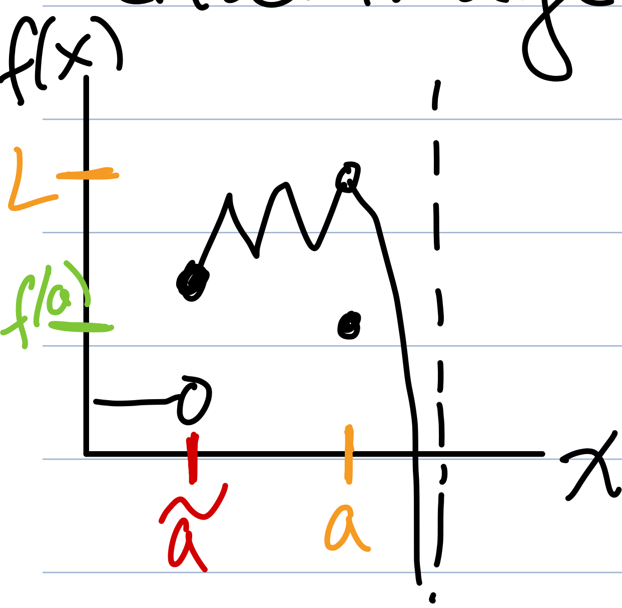
Theorem: Given a function f and $a \in \mathbb{R}$,

$\lim_{x \to a} f(x)$ exists $\iff$ both one sided limits exist and $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$

If either equivalent condition is true, then

$$\lim_{x \to a} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x).$$

Mental image



note: $L \neq f(a)$

$$\lim_{x \to a} f(x) = L = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$$

$$\lim_{x\to \tilde{a}^-} f(x) \neq \lim_{x\to\tilde{a}^+} f(x)$$

and the two sided limit D.N.E.

Pf: First, suppose $\lim_{x\to a} f(x)$ exists, that is,

$$\lim_{x\to a} f(x) = L$$

for some $L \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$. Thus, there exists (a_0, a_1) s.t. $a \in (a_0, a_1)$, $(a_0, a_1) \setminus \{a\} \subseteq \mathrm{dom}(f)$, and for any sequence x_n s.t. $x_n \in (a_0, a_1) \setminus \{a\}$ and $x_n \to a$, we have $\lim_{n\to\infty} f(x_n) = L$.

Then, we have that $(a_0, a) \subseteq \mathrm{dom}(f)$ and $(a, a_1) \subseteq \mathrm{dom}(f)$. For any sequence x_n s.t. either

$x_n \in (a_0, a)$ or $x_n \in (a, a_1)$ satisfying $x_n \to a$, our assumption on existence of two sided limit ensures $\lim_{n\to\infty} f(x_n) = L$.

Thus, both one sided limit exist and their values coincide with the two sided limit.

For the opposite implication assume there exists $L \in \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ s.t. $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$.

Thus, $\exists\ (a_0, a) \subseteq dom(f)$ and $(a, a_1) \subseteq dom(f)$ s.t.

for every sequence x_n satisfying $x_n \to a$ and either $x_n \in (a_0, a)$ or $x_n \in (a, a_1)$, we have $\lim_{n\to\infty} f(x_n) = L$.

Then $(a_0, a_1) \setminus \{a\} \subseteq \mathrm{dom}(f)$ and $a \in (a_0, a_1)$. Suppose $x_n \in (a_0, a_1) \setminus \{a\}$ and $x_n \to a$. We must show $\lim_{n\to\infty} f(x_n) = L$. Fix an arbitrary subsequence x_{n_k}.

Since x_{n_k} must either have infinitely many elements less than a ~~or~~ infinitely many elements greater than a, there exists a further subsequence $x_{n_{k_\ell}}$ s.t. either $x_{n_{k_\ell}} \in (a_0, a)$ $\forall\, \ell \in \mathbb{N}$

or $x_{n_{k_\ell}} \in (a, a_1)$ $\forall \ell \in \mathbb{N}$.

In either case, $\lim_{\ell \to \infty} f(x_{n_{k_\ell}}) = L$.

Thus, $\lim_{n \to \infty} f(x_n) = L$.

<u>Recall</u>: Suppose $L \in \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$.
$\lim_{n\to\infty} z_n = L \Longleftrightarrow$ Every subsequence z_{n_k} of z_n has a further subsequence $z_{n_{k_\ell}}$ s.t. $\lim_{\ell\to\infty} z_{n_{k_\ell}} = L$.

$\square$

Recall: Series (HW6) $a_k \in \mathbb{R}, \forall k \in \mathbb{N}$

Def: Given a series $\sum_{k=1}^{\infty} a_k$,

define the partial sum sequence by $S_n := \sum_{k=1}^{n} a_k$.

Then $\sum_{k=1}^{\infty} a_k$ converges to $L \in \mathbb{R}$

if $\lim_{n \to \infty} S_n = L$. Likewise, the series diverges to $\pm\infty$ if S_n diverges to $\pm\infty$.

Recall:

Cor: If $\sum_{k=1}^{\infty} a_k$ converges,

then $\lim_{k \to \infty} a_k = 0$.

Lem: Consider $\sum_{k=1}^{\infty} r^k$ for $r \in \mathbb{R}$.

If $|r|<1$, $\sum_{k=1}^{\infty} r^k = \frac{1}{1-r}$

If $|r|>1$, the series does not converge.

Another easy fact about series...

Prop (Comparison test):
If $|b_k| \le a_k$ $\forall k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=1}^{\infty} b_k$ converges.

Pf: Let $s_n := \sum_{k=1}^{n} a_k$, $t_n = \sum_{k=1}^{n} b_k$.
We must show t_n converges.

Note that, for $m \leq n$

$$|t_n - t_m| = \left| \sum_{k=m+1}^{n} b_k \right|$$

$$\leq \sum_{k=m+1}^{n} |b_k|$$

$$\leq \sum_{k=m+1}^{n} a_k$$

$$= |s_n - s_m|$$

Since s_n is convergent, hence Cauchy, $\forall\, \varepsilon > 0$, $\exists$ M s.t. $n \geq m \geq M$ ensures $|s_n - s_m| < \varepsilon \Rightarrow |t_n - t_m| < \varepsilon$.

Thus t_n is Cauchy, hence convergent. $\square$