Lecture 17

CS 117, S25

Announcements:

- HW7 due Friday
- Final Exam Monday 12-3pm

Recall:

Def: Given a function f, $a \in \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$, we say that

Case 1: $a \in \mathbb{R}$

$$\lim_{x \to a^+} f(x) = L$$

Case 2: $a = -\infty$

$$\lim_{x \to -\infty} f(x) = L$$

"x approaches a from above"

Case 1: $a \in \mathbb{R}$

$$\lim_{x \to a^-} f(x) = L$$

Case 2: $a = +\infty$

$$\lim_{x \to +\infty} f(x) = L$$

"x approaches a from **below**"

if there exists a (nonempty) interval

$I = (a, a_1)$, $a_1 \in \mathbb{R}$ || $I = (a_0, a)$, $a_0 \in \mathbb{R}$

s.t. $I \subseteq \mathrm{dom}(f)$ and for every sequence $x_n \in I$ s.t. $\lim_{n \to \infty} x_n = a$, we have $\lim_{n \to \infty} f(x_n) = L$.

Theorem: Given a function f and $a \in \mathbb{R}$,

$\lim_{x \to a} f(x)$ exists $\iff$ both one sided limits exist and $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$

If either equivalent condition is true, then

$$\lim_{x \to a} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x).$$

Last topic of course: sequences and series of functions

__Motivation__: In 118, you will prove the __Stone Weierstrass Theorem__, which ensures that every continuous fn on an interval $[a,b]$ may be approximated as closely as possible by a polynomial.

This result is important in...

- __applications__
  computational (engineering)
  theoretical (physics)

- __theory__
  how can we define functions such as $\sin(x)$, $\cos(x)$, e^x, ...?

Mental image

$n=1$

$f(x)=\sin(x)$

$n=3$

What do we mean by "approximated as closely as possible"?

Def: Given $S \subseteq \mathbb{R}$, a sequence of functions $f_n: S \to \mathbb{R}$ converges pointwise to a function $f: S \to \mathbb{R}$ if

$$\lim_{n\to\infty} f_n(x) = f(x), \qquad \forall x \in S$$

We will often write $f_n \to f$ pointwise (on S).

Ex: Consider $S = [0,1]$ and $f_n(x) = x^n$. What is the pointwise limit?

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}$$

By HW3, Q5, $f_n \to f$ pointwise

Wait - weren't we trying to approximate cts fns by polynomials?

It turns out that pointwise convergence of cts fns (e.g. polynomials) is not strong enough to ensure the limit is cts.

How to strengthen it?

Observe that

$f_n \to f$ pointwise on S

$$\Updownarrow$$

$\forall x \in S,\ \forall \varepsilon > 0,\ \exists\ N$ s.t. $n > N$ ensures $|f_n(x) - f(x)| < \varepsilon$.

$$\Updownarrow$$

$\forall \varepsilon > 0,\ \forall x \in S,\ \exists\ N$ s.t. $n > N$ ensures $|f_n(x) - f(x)| < \varepsilon$.

<u>Def</u>: Given $S \subseteq \mathbb{R}$, a sequence of functions $f_n : S \to \mathbb{R}$ <u>converges uniformly</u> to a function $f : S \Rightarrow \mathbb{R}$ if, $\forall \varepsilon > 0$, $\exists\ N$ s.t. $n > N$ ensures

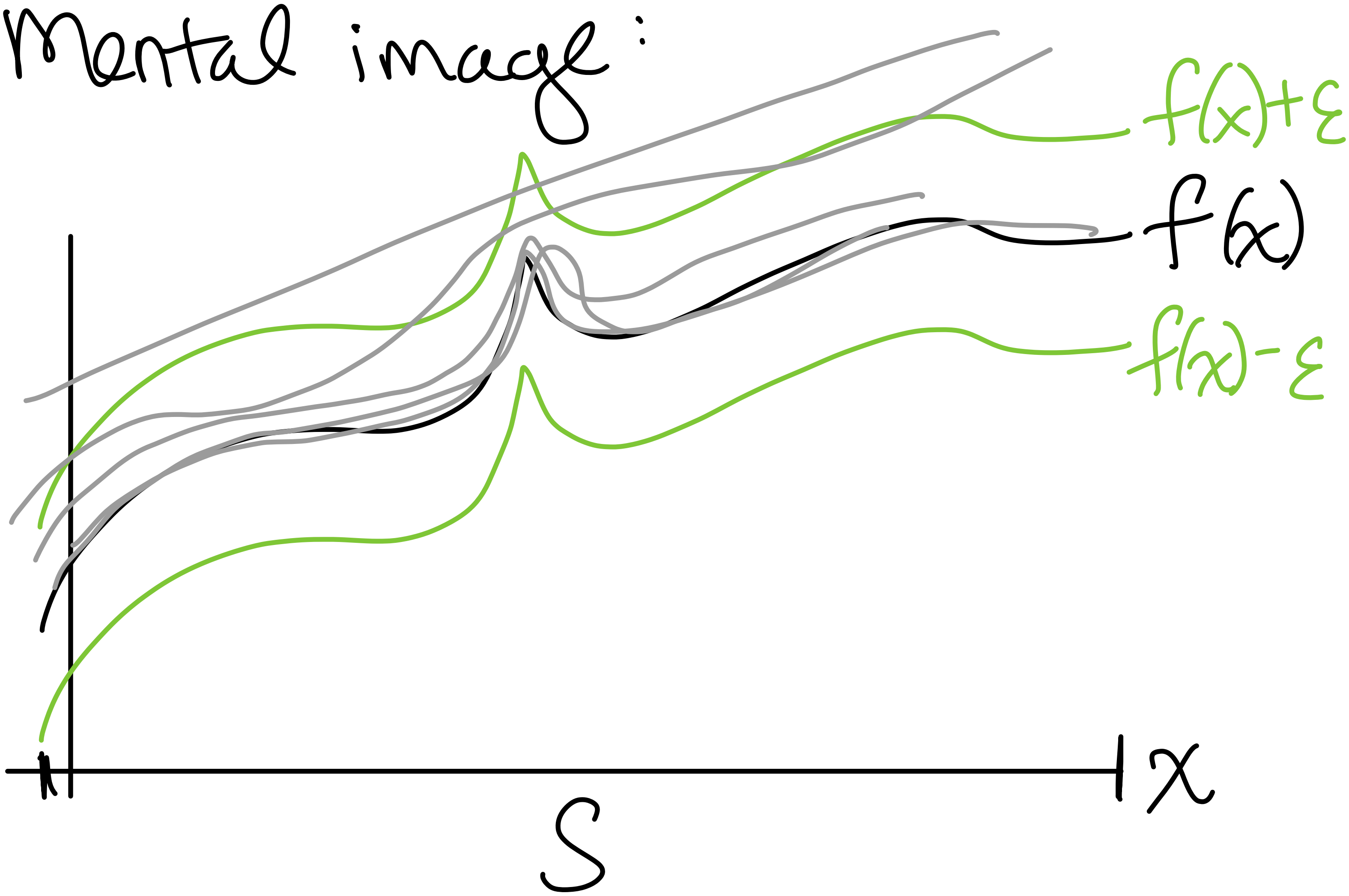
$$|f_n(x) - f(x)| < \varepsilon, \quad \forall x \in S.$$

$$\Updownarrow$$

$$f(x) - \quad < f_n(x) < f(x) + \varepsilon$$

We write $f_n \to f$ uniformly (on S).

Mental image:



<u>Rmk</u>: The defn of uniform convergence is reminiscent of the fact that, to be <u>uniformly continuous</u> on S, $\forall \varepsilon > 0$, $\exists\ \delta > 0$ s.t. $x, y \in S$, $|x-y| < \delta$ ensures $|f(x) - f(y)| < \varepsilon$.

(the same δ must work for all $x, y \in S$.)

Thm (uniform limit of cts fns is cts) Given $S \subseteq \mathbb{R}$ and $f_n, f : S \to \mathbb{R}$ s.t. $f_n \to f$ uniformly on S, suppose $\mathrm{dom}(f) = S$.

Then $\forall x_0 \in S$, if f_n is cts at x_0 $\forall n \in \mathbb{N}$, f is cts at x_0.

Rmk: It follows that, under the hypotheses of the theorem, if f_n is cts on S, then its uniform limit f is cts on S.

Ex: Why must we assume $\mathrm{dom}(f)=S$?

Consider $S=[0,1]$, $f_n(x)=\frac{1}{n}$, $f(x)=0$. Then $f_n \to f$ on S uniformly, and f_n, f are all cts.

However, we could also have $\tilde{f}(x)=\begin{cases} 0 & \text{if } x\in[0,1] \\ 1 & \text{if } x\notin[0,1] \end{cases}$.

Then $f_n \to f$ uniformly on S, but $\tilde{f}$ is not continuous at $1\in S$.

Pf: Fix $x_0 \in S$ arbitrary and assume f_n is cts at x_0 $\forall$ $n \in \mathbb{N}$. We will show f is cts at x_0, via ε-δ definition.

Fix $\varepsilon > 0$ arbitrary. Since $f_n \to f$ unif, $\exists$ N_0 s.t. $n > N_0$ ensures

$$(*) \quad |f_n(x) - f(x)| < \frac{\varepsilon}{3}, \; \forall x \in S.$$

Choose $N_1 \in \mathbb{N}$ s.t. $N_1 > N_0$. Since f_{N_1} is cts at x_0, $\exists$ $\delta > 0$ s.t.

$$(*) \quad x \in S, \; |x - x_0| < \delta \text{ ensures } |f_{N_1}(x) - f_{N_1}(x_0)| < \frac{\varepsilon}{3}.$$

Thus, $x \in S$, $|x - x_0| < \delta$ ensures

$$
\begin{aligned}
|f(x) - f(x_0)| &= |f(x) - f_{N_1}(x) + f_{N_1}(x) - f_{N_1}(x_0) \\
&\qquad + f_{N_1}(x_0) - f(x_0)| \\
&\leq \underbrace{|f(x) - f_{N_1}(x)|}_{(*)} + \underbrace{|f_{N_1}(x) - f_{N_1}(x_0)|}_{(*)} \\
&\qquad + \underbrace{|f_{N_1}(x_0) - f(x_0)|}_{(*)} \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}
$$

$\square$

==== END OF MATERIAL FOR FINAL EXAM ====

Recall: Series (HW6)

$a_k \in \mathbb{R}, \forall k \in \mathbb{N}$

Def: Given a series $\sum_{k=1}^{\infty} a_k$,

define the partial sum sequence by $S_n := \sum_{k=1}^{n} a_k$.

Then $\sum_{k=1}^{\infty} a_k$ converges to $L \in \mathbb{R}$

if $\lim_{n\to\infty} S_n = L$. Likewise, the series diverges to $\pm\infty$ if S_n diverges to $\pm\infty$.

Cor: If $\sum_{k=1}^{\infty} a_k$ converges,

then $\lim_{k\to\infty} a_k = 0$.

Lem: Consider $\sum_{k=0}^{\infty} r^k$ for $r \in \mathbb{R}$.

If $|r|<1$, $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ thank you, Chenxi!

If $|r| \geq 1$, the series does not converge.

Prop (Comparison test):
If $|b_k| \leq a_k \ \forall k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=1}^{\infty} b_k$ converges.

Def: $\sum_{k=1}^{\infty} a_k$ converges absolutely if $\sum_{k=1}^{\infty} |a_k|$ converges

Cor: absolute convergence $\Rightarrow$ convergence

We will now apply these to prove a familiar fact...

Thm (Root test): Given a series $\sum_{k=1}^{\infty} a_k$, define

$$\alpha := \limsup_{k\to\infty} |a_k|^{1/k} = \lim_{K\to\infty} \underbrace{\sup\{|a_k|^{1/k} : k > K\}}_{y_K}$$

Then,

- if $\alpha < 1$, the series converges absolutely
- if $\alpha > 1$, the series diverges

Rmk: If $\alpha = 1$, either convergence or divergence is possible

Pf: First, suppose $\alpha < 1$. Fix $\varepsilon > 0$ s.t. $\alpha + \varepsilon < 1$. By definition of y_K $\exists$ K_0 s.t. $K > K_0$ ensures

$$\alpha - \varepsilon < y_K < \alpha + \varepsilon.$$

In particular, choosing $K_1 \in \mathbb{N}$, $K_1 > K_0$, we have

$$\sup\{|a_k|^{1/k} : k > K_1\} = y_{K_1} < \alpha + \varepsilon$$

Thus, $\forall k > K_1$, $|a_k|^{1/k} < \alpha + \varepsilon$.

$$\Downarrow$$

$$|a_k| < \underbrace{(\alpha + \varepsilon)}_{r}{}^k$$

Since $\sum_{k=1}^{\infty} (\alpha+\varepsilon)^k$ is a geometric series with $r = \alpha + \varepsilon \in (0,1)$, it converges. Thus $\sum_{k=1}^{\infty} |a_k|$ converges, by the comparison test. Hence $\sum_{k=1}^{\infty} a_k$ converges absolutely.

Now, suppose $\alpha > 1$. Since the limsup is the largest subsequential limit, there exists a subsequence of $|a_k|^{1/k}$ with limit $\alpha > 1$. Thus $|a_k|^{1/k} > 1$ for infinitely many $k \in \mathbb{N}$, so $|a_k| > 1^k = 1$ for infinitely many $k \in \mathbb{N}$. Thus $\lim_{k \to \infty} a_k \neq 0$, so the series diverges. $\square$