Lecture 18

CS 117, S25

Announcements:

- HW7 due Friday
- Final Exam Monday 12-3pm

<u>Recall:</u>

<u>Def</u>: Given $S \subseteq \mathbb{R}$, a sequence of functions $f_n : S \to \mathbb{R}$ <u>converges pointwise</u> to a function $f: S \to \mathbb{R}$ if

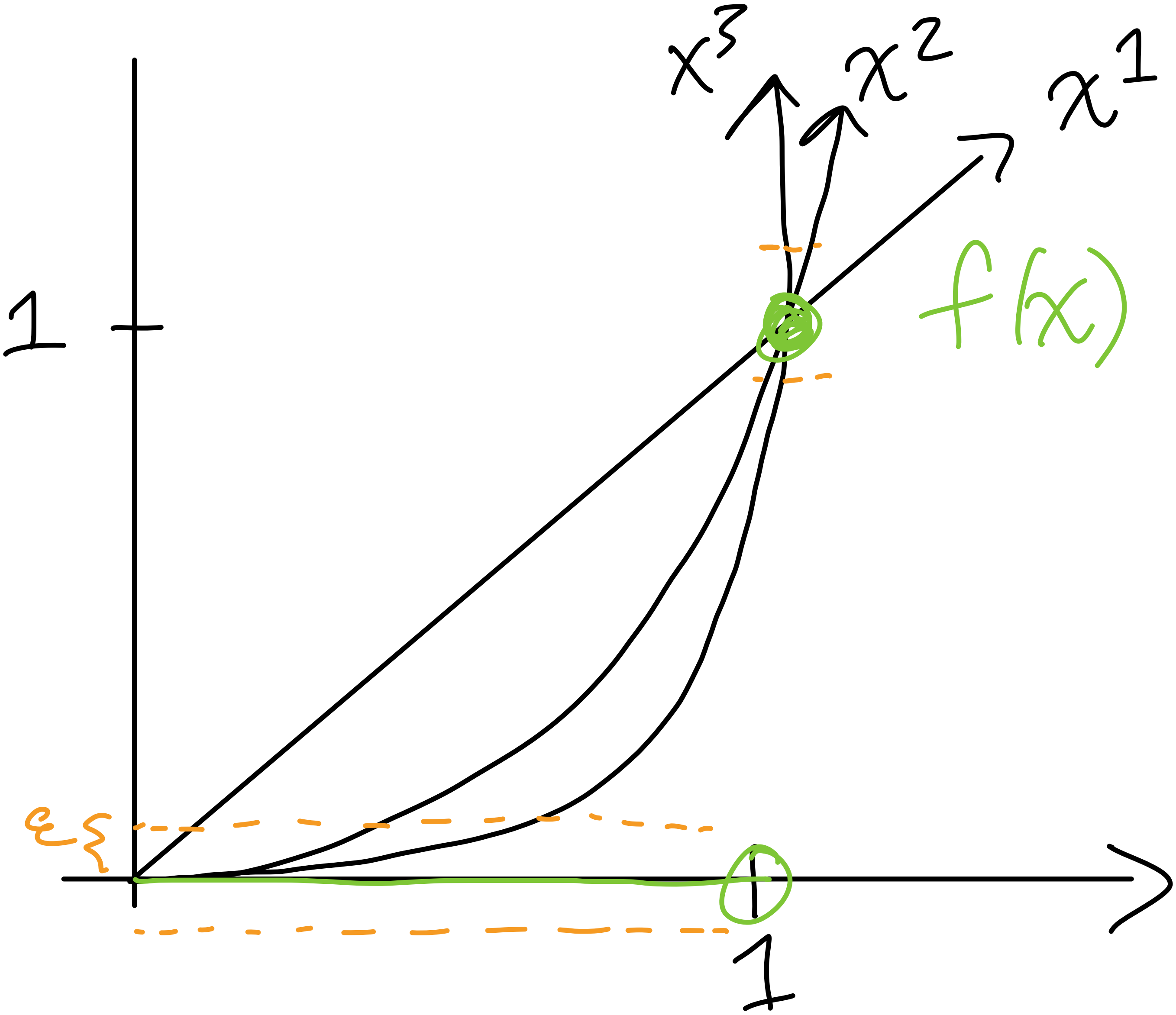
$$\lim_{n \to \infty} f_n(x) = f(x), \qquad \forall x \in S$$

Def: Given $S \subseteq \mathbb{R}$, a sequence of functions $f_n : S \to \mathbb{R}$ converges uniformly to a function $f : S \to \mathbb{R}$ if, $\forall \varepsilon > 0$, $\exists N$ s.t. $n > N$ ensures

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall x \in S.$$

Thm: (uniform limit of cts fns is cts) Given $S \subseteq \mathbb{R}$ and $f_n, f : S \to \mathbb{R}$ s.t. $f_n \to f$ uniformly on S, suppose $\mathrm{dom}(f) \supseteq S$.

Then $\forall x_0 \in S$, if f_n is cts at x_0 $\forall n \in \mathbb{N}$, f is cts at x_0.



$$f_n(x) = x^n$$

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}$$

Rmk: It follows that, under the hypotheses of the theorem, if f_n is cts on S, then its uniform limit f is cts on S.

END OF MATERIAL FOR FINAL EXAM

Thm (Root test): Given a series $\sum_{k=1}^{\infty} a_k$, define

$$\alpha := \limsup_{k\to\infty} |a_k|^{1/k} = \lim_{K\to\infty} \underbrace{\sup\{|a_k|^{1/k} : k > K\}}_{y_K}.$$

Then,

- if $\alpha < 1$, the series converges absolutely
- if $\alpha > 1$, the series diverges

Series of functions

Important example: Power series

$$\sum_{n=0}^{\infty} a_n x^n$$

("infinitely long" polynomial)

When does a power series approximate a continuous function?

When does it approximate a real valued function, i.e., when does the series converge?

Thm: Given a power series $\sum_{n=0} a_n x^n$,

if $\beta = 0, +\infty$
$R = +\infty, 0$.

define $\beta := \limsup |a_n|^{1/n}$, $R := \frac{1}{\beta}$.
Then
- the series converges for $|x| < R$
- the series diverges for $|x| > R$.

Rmk: R is called the radius of convergence of the power series.

Rmk: Note that all power series converge at $x = 0$.

Pf: Applying the root test

$$\alpha := \limsup_{n\to\infty} |a_n x^n|^{1/n}$$

$$= \limsup_{n\to\infty} |a_n|^{1/n}|x|$$

$$= \lim_{N\to\infty} \sup \underbrace{\{|a_n|^{1/n}|x| : n > N\}}_{|x|S}$$

$$= |x| \lim_{N\to\infty} \sup\{|a_n|^{1/n} : n > N\}$$

$$= |x| \limsup_{n\to\infty} |a_n|^{1/n}$$

$$= |x|\beta$$

Thus $|x| < R \Rightarrow \alpha < 1 \Rightarrow$ convergence

$|x| > R \Rightarrow \alpha > 1 \Rightarrow$ divergence $\square$

Ex: Consider $\sum_{n=0}^{\infty} x^n$.

Then $\beta = \limsup_{n\to\infty} 1^{1/n} = 1$.

The previous theorem ensures the series converges for $|x|<1$ and diverges for $|x|>1$.

pointwise ✓

In fact, for $|x|<1$, it converges to $f(x) = \frac{1}{1-x}$.

Given that we want to approximate <u>cts</u> fns by power series, we are interested when they converge uniformly, that is when the partial sum sequence

$$f_N(x) := \sum_{n=0}^{N} a_n x^n$$

converges uniformly.

For sequences s_n of real numbers, s_n convergent $\Leftrightarrow$ s_n Cauchy.

In fact, the same is true for sequences of fns wrt uniform convergence.

Def: Given $S \subseteq \mathbb{R}$, a sequence of functions $f_n : S \to \mathbb{R}$ is <u>uniformly</u> <u>Cauchy</u> on S if

$\forall\ \varepsilon>0,\ \exists\ N$ s.t. $m,n>N$ ensures $|f_n(x)-f_m(x)|<\varepsilon,\ \forall x\in S.$

Thm: Given $S\subseteq\mathbb{R}$, a sequence of functions $f_n: S\to\mathbb{R}$ that is uniformly Cauchy, then there exists $f: S\to\mathbb{R}$ s.t. $f_n\to f$ uniformly on S.

Pf: First, we must "guess" the limit f.

Since f_n is uniformly Cauchy, for any fixed $x\in S$, $f_n(x)$ is a Cauchy sequence of real numbers. Hence it converges. Denote the limit by $f(x)$.

We will now show $f_n \to f$ uniformly on S.

Fix $\varepsilon > 0$. Since f_n is uniformly Cauchy, $\exists N$ s.t. $m, n > N$ ensures

$$|f_n(x) - f_m(x)| < \frac{\varepsilon}{2}, \forall x \in S.$$

$$\Downarrow$$

$$f_m(x) - \frac{\varepsilon}{2} < f_n(x) < f_m(x) + \frac{\varepsilon}{2}$$

Sending $m \to \infty$, for $n > N$,

$$f(x) - \frac{\varepsilon}{2} \leq f_n(x) \leq f(x) + \frac{\varepsilon}{2}, \forall x \in S.$$

$$\Downarrow$$

$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{2}$$

Thus, for $n > N$,
$$|f_n(x) - f(x)| < \varepsilon, \ \forall x \in S.$$
$\square$

When is a power series converging uniformly, so we know its limit is cts?

Thm: Given a power series
$$\sum_{n=0}^{\infty} a_n x^n$$
with radius of convergence $R > 0$, $\forall R_1 \in (0, R)$, the power series converges uniformly to a continuous function on $[-R_1, R_1]$.

Pf: By the previous theorem, it suffices to show

$$f_N(x) := \sum_{n=0}^{N} a_n x^n$$

is uniformly Cauchy on $[-R_1, R_1]$.

Fix $\varepsilon > 0$. Since $\sum_{n=0}^{\infty} a_n x^n$

converges absolutely on $(-R, R)$, the series of real numbers

$$\sum_{n=0}^{\infty} |a_n| R_1^n$$

converges. Thus, the corresponding partial sum sequence

$$s_N := \sum_{n=0}^{N} |a_n| R_1^n$$

is Cauchy.

Thus, $\forall\, \varepsilon > 0$, $\exists\, N_0$ s.t. $N, M > N_0$ ensures

WLOG $N \geq M$

$$\sum_{n=M+1}^{N} |a_n||x|^n \leq \sum_{n=M+1}^{N} |a_n| R_1^n = |s_N - s_M| < \varepsilon$$

for all $x \in [-R_1, R_1]$

$$\| \quad |f_N(x) - f_M(x)|.$$

This shows $f_N(x)$ is uniformly Cauchy on $[-R_1, R_1]$. $\square$

A result that is believable but we will not prove...

<u>Thm</u>: Given $f(x) := \sum_{n=0}^{\infty} a_n x^n$ with radius of convergence $R > 0$, then f is differentiable on $(-R, R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Important example of a power series that converges uniformly on $[-R_1, R_1]$ for ~~all~~ all $R_1 > 0$, but we don't have nice formula for limit...

Ex: $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

Radius of convergence depends on...

$$\beta := \limsup_{n\to\infty} \left|\frac{1}{n!}\right|^{1/n}$$

We will show $\beta = 0$, so $R = +\infty$.
It suffices to show

$$(n!)^{1/n} \to +\infty.$$

Observation:
$(n!)^2 = [1 \cdot n][2(n-1)] \cdot \ldots \cdot [(n-1) \cdot 2][n \cdot 1]$
Each of the n products
is of the form

$(k+1)(n-k)$ for $0 \le k \le n$

Each of these is $\ge n$. Thus

$$(n!)^2 \ge n^n$$

$$\Updownarrow$$

$$(n!)^{1/n} \ge \sqrt{n}$$

We can obtain familiar properties of exponential directly from series...

$$\frac{d}{dx} e^x = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{d}{dx} \frac{x^n}{n!}$$

$$= \sum_{n=1}^{\infty} n \frac{x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = e^x$$