Lecture 1

CS 117, S25

Course goal: transition to higher level math

- "What is a proof?" ⇒ "Let's prove interesting things"
- This is a mathematical writing course.
  - ↳ You must back up your claims using clear, logical arguments.
  - ↳ You must be able to precisely state important definitions and theorems.
- If something doesn't make sense...
  1. Carefully read all relevant definitions and theorems. Get the textbook!
  2. Come to office hours.
  3. Hang in there. If you stay on top of learning definitions and theorems, things will start to make sense. If you don't, things will become more confusing.

Q: Why analysis? What is analysis?

A: It takes everything you learned in Calculus and puts it on rigorous mathematical footing.

A': Analysis is the mathematics of <u>approximation</u>, allowing us to...

- ▫ quantify accuracy of mathematical models with respect to data
- ▫ study convergence behavior in the limit of
  - more data
  - more computational power
- ▫ estimate the likelihood and sensitivity of mathematical predictions

Numbers!

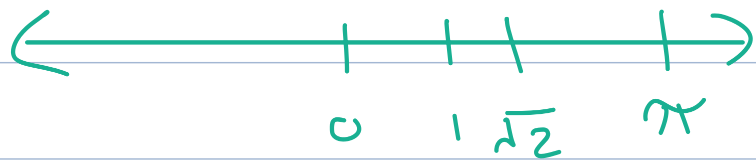
Natural numbers $\mathbb{N} = \{1, 2, 3, 4, \dots\}$

Integers $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

Rational numbers $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z},\ n \neq 0\}$

Real numbers $\mathbb{R} = ?$

Intuitively, $\mathbb{R}$ is all the numbers on the number line...



Goal: define $\mathbb{R}$.

Def: A <u>binary operation</u> on a set X is a function from $X \times X$ to X.

Def: (field) A set F is a <u>field</u> if it has two binary operations (addition and multiplication) that satisfy the following properties $\forall a, b, c \in F$:

(A1) $a+(b+c)=(a+b)+c$ associativity

(A2) $a+b=b+a$ commatativity

(A3) $\exists!$ element $0\in F$ s.t. identity

$\forall a\in F$, $a+0=a$.

(A4) for each $a\in F$, $\exists!\, b\in F$ inverse

s.t. $a+b=0$; denote $-a:=b$

(m1) $a(bc)=(ab)c$ associativity

(m2) $ab=ba$ commatativity

(m3) $\exists!$ element $1\in F\setminus\{0\}$ identity

s.t., $\forall a\in F$, $a\cdot 1=a$

(m4) for each $a\in F\setminus\{0\}$, $\exists!$ inverse

$b\in F$ s.t. $ab=1$;

denote $\frac{1}{a}=a^{-1}=b$.

(D1) $a(b+c)=ab+ac$ distributive law

Remark: $\mathbb{N}$ and $\mathbb{Z}$ aren't fields

Thm: $\mathbb{Q}$ is a field.

Using the definition of a field, you can rigorously prove familiar algebraic properties.

<u>Thm</u>: If F is a field, then $\forall a,b \in F$,
(i) if $a+c=b+c$, then $a=b$;
(ii) $a \cdot 0 = 0$.

<u>Pf</u>: First, we show (i). By (A4), $\exists -c$ s.t. $c+(-c)=0$. Thus,

$$
\begin{aligned}
a+c=b+c &\Rightarrow a+c+(-c) = b+c+(-c) \\
&\overset{(A1)}{\Longrightarrow} a+(c-c) = b+(c-c) \\
&\overset{(A4)}{\Longrightarrow} a+0 = b+0 \\
&\overset{(A3)}{\Longrightarrow} a=b.
\end{aligned}
$$

We now show (ii). By (A3), $0+0=0$, so $\forall a \in F$,

$$
\begin{aligned}
a\cdot(0+0) = a\cdot 0 &\overset{(D2)}{\Longrightarrow} a\cdot 0 + a\cdot 0 = a\cdot 0 \\
&\overset{(A3)}{\Longrightarrow} a\cdot 0 + a\cdot 0 = a\cdot 0 + 0 \\
&\overset{(A2)}{\Longrightarrow} a\cdot 0 + a\cdot 0 = 0 + a\cdot 0
\end{aligned}
$$

Thus, by (i), $a \cdot 0 = 0$. $\square$

The definition of a field captures our intuition of how elements in $\mathbb{R}$ should "interact" with each other via addition and multiplication.

An equally important feature of $\mathbb{R}$ is that we perceive its elements as possessing an "order," from left to right, on the number line.

Def (ordered field): A field F is an <u>ordered field</u> if it has an ordering relation $\leq$ so that, for all $a, b, c \in F$,

(O1) either $a \leq b$ or $b \leq a$ totality
(O2) if $a \leq b$ and $b \leq a$, then $a = b$ antisymmetry
(O3) if $a \leq b$ and $b \leq c$, then $a \leq c$ transitivity
(O4) if $a \leq b$, then $a + c \leq b + c$ addition
(O5) if $a \leq b$ and $c \geq 0$, then $ac \leq bc$ multiplication

Def: Given an ordered field F and $a, b \in F$, if $a \leq b$ and $a \neq b$, then write $a < b$.

Using the definition of an ordered field, we can obtain many familiar rules about inequalities.

Thm: Suppose F is an ordered field. Then $\forall a, b, c \in F$,

(i) $a \leq b \Rightarrow -b \leq -a$

(ii) $a \leq b$ and $c \leq 0 \Rightarrow ac \geq bc$

(iii) $0 \leq a$ and $0 \leq b \Rightarrow 0 \leq ab$.

(iv) $0 \leq a^2$, where $a^2 := a \cdot a$

(v) $0 < a \Rightarrow 0 < \frac{1}{a}$.

Pf: We will show (i) and (iii).

To see (i), suppose $a \le b$. By (O4), $a + (-a-b) \le b + (-a-b)$. By (A1-4), $-b \le -a$.

To see (iii), suppose $0 \le a$ and $0 \le b$, by (O5), $\underbrace{0 \cdot b}_{=0,\text{ by previous thm}} \le a \cdot b$. $\square$

Rmk: For any ordered field, $0 < 1$.

Thm: $\mathbb{Q}$ is an ordered field.

Rmk: We will show on the homework that any ordered field F has a subfield that is isomorphic to $\mathbb{Q}$.

Ex: $\left\{ \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix} : q \in \mathbb{Q} \right\}$ is an ordered field.

Rmk: We will show on the homework that any ordered field F has a subfield that is isomormphic to $\mathbb{Q}$.

An important property of an ordered field is that...

Prop: Suppose F is an ordered field. Then $\forall\, p, q \in F$ with $p<q$, $\exists\, r \in F$ s.t. $p<r<q$.

Pf: Let $2 := 1+1$. Since $0<1$, $\underbrace{0+1}_{1} < \underbrace{1+1}_{2}$, so $0<2$. By Thm(v), $0<\frac{1}{2}$.

Since $p < q$, $\frac{p}{2} < \frac{q}{2}$.

By the distributive law,

$$p + p = (1+1)p = 2p \Rightarrow \frac{p+p}{2} = p.$$

Therefore,

$$p = \frac{p+p}{2} \overset{(D2)}{=} \frac{p}{2} + \frac{p}{2} < \underbrace{\frac{p}{2} + \frac{q}{2}}_{r :=} < \frac{q}{2} + \frac{q}{2} = q$$

$\square$

On any ordered field F, we may define a notion of absolute value.

Def: For any $a \in F$, $|a| := \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$

Thm (basic properties of $|\cdot|$): For all $a, b \in F$,

(i) $|a| \geq 0$

(ii) $|ab| = |a||b|$ ← absolute value distributes over multiplication

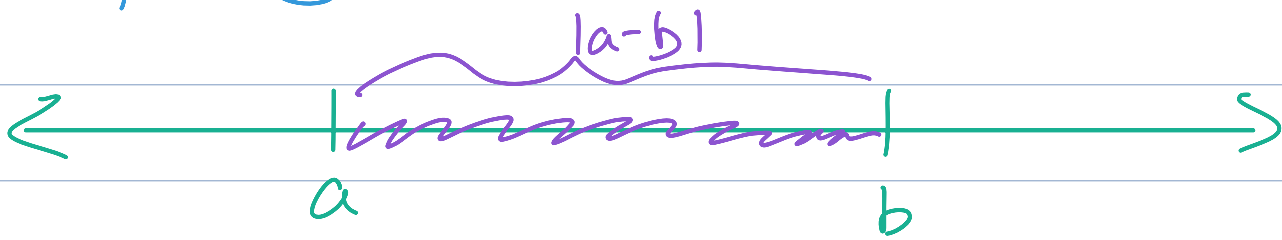
(iii) $|a| \geq a$ and $|a| \geq -a$

(iv) $|a+b| \leq |a| + |b|$ ← Triangle inequality

Pf: Homework.

We can use the absolute value to define a notion of <u>distance</u> between any two elements of an ordered field.

Def: For any $a, b \in F$,
$\mathrm{dist}(a,b) = |a-b|$.



Likewise, on an ordered field, we can define what it means for a set to be bounded above or below.

What about when a set "almost" has a maximum?