Lecture 3

CS 117, S25

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Announcements:

- No class next week (Tues Apr 15, Thur Apr 17)

Recall:

Def (maximum, minimum): Suppose $S \subseteq F$, where F is an ordered field.

- If there exists $s_0 \in S$ satisfying $s_0 \geq s \ \forall s \in S$, then s_0 is the <u>maximum</u> of S and write $s_0 = \max(S)$.
  "s_0 is the largest element in the set"

- If there exists $s_0 \in S$ satisfying $s_0 \leq s \ \forall s \in S$, then s_0 is the <u>minimum</u> of S and write $s_0 = \min(S)$.
  "s_0 is the smallest element in the set"

Def: (bounded above/below): Suppose $S \subseteq F$, for an ordered field F,

- If there exists $M \in F$ s.t. $s \leq M$ $\forall s \in S$, then S is <u>bounded above</u> and M is an <u>upper bound</u> of S.

- If there exists $m \in F$ s.t. $s \geq m$ $\forall s \in S$, then S is <u>bounded below</u> and m is an <u>lower bound</u> of S.

- If S is bounded above <u>and</u> below, then S is <u>bounded</u>.

Def (supremum/infimum): Consider an ordered field F.

- If $S \subseteq F$ is bounded above and there exists $M_0 \in F$ satisfying...

(a) M_0 is an upper bound of S
(b) if M is an upper bound of S, then $M_0 \leq M$

we say M_0 is the <u>supremum</u> of S and write $M_0 = \sup(S)$.
"M_0 is the least upper bound of S"

- If $S \subseteq F$ is bounded below and there exists $m_0 \in F$ satisfying...
  (a) m_0 is a lower bound of S
  (b) if m is a lower bound of S, then $m_0 \geq m$.

we say m_0 is the <u>infimum</u> of S and write $m_0 = \inf(S)$.
"m_0 is the greatest lower bound of S"

Thm: Given $S \subseteq F$, F an ordered field,

- if $\max(S)$ exists, $\sup(S) = \max(S)$;
- if $\min(S)$ exists, $\inf(S) = \min(S)$

Def (real numbers): The real numbers is the ordered field containing $\mathbb{Q}$ with the property that every nonempty subset $S \subseteq \mathbb{R}$ that is bounded above has a supremum.

"the least upper bound property of $\mathbb{R}$"

Thm: The real numbers exist and are unique.

Remark: $\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$

strictly contained in

in a previous course, you saw $\sqrt{2} \notin \mathbb{Q}$; on HW, you will show $\sqrt{2} \in \mathbb{R}$

Thm: The natural numbers $\mathbb{N}$ is the smallest subset of $\mathbb{R}$ having the properties that

(i) $1 \in \mathbb{N}$

(ii) $n \in \mathbb{N} \Rightarrow n+1 \in \mathbb{N}$

"smallest", in the sense of set inclusion.

Rmk: This property of $\mathbb{N}$ forms the basis for proof by induction.

We'll study two major theorems for $\mathbb{R}$:

Archimedean Property

$\mathbb{Q}$ is dense in $\mathbb{R}$

MAJOR THM #1

Thm (Archimedean Property): If $a, b \in \mathbb{R}$ satisfy $a > 0$ & $b > 0$, then $\exists\, n \in \mathbb{N}$ s.t. $na > b$. ← bathtub

spoon

"even with a very small spoon, you can fill a large bathtub"

Pf: Fix $a, b \in \mathbb{R}$ with $a > 0$ and $b > 0$.

Assume that, $\forall n \in \mathbb{N}$, we have $na \leq b$.

Define $S = \{na : n \in \mathbb{N}\}$. By hypothesis, b is an upper bound of S. Thus, by defn of $\mathbb{R}$, S has a supremum. Define $s_0 := \sup(S)$.

can't be an upper bound

Since $a > 0$, we have $\underbrace{s_0 - a}_{} < s_0 < s_0 + a$.
Since $s_0 = \sup(S)$, there exists $n_0 \in \mathbb{N}$ s.t. $s_0 - a < n_0 a$.
Rearranging the inequality gives $s_0 < (n_0 + 1)a \in S$, contradicting that s_0 is an upper bound. $\square$

As a consequence of the Archimedean Property, we have a few useful lemmas...

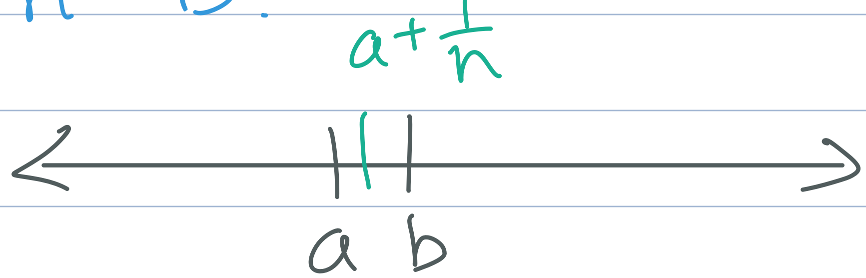
<u>Lemma</u>: For any $x \in \mathbb{R}$, $\exists\, n \in \mathbb{N}$ s.t. $x < n$.

<u>Pf</u>: First, suppose $x > 0$. Then, by Arch. Prop. for $b = x > 0$ and $a = 1 > 0$, $\exists\, n \in \mathbb{N}$ s.t. $n > x$.

Now suppose $x \leq 0$. Then $x \leq 0 < 1$, so the result holds for $n = 1 \in \mathbb{N}$.

<u>Lemma</u>: For $a, b \in \mathbb{R}$, $a < b$, $\exists\, n \in \mathbb{N}$ s.t. $a + \frac{1}{n} < b$.

Mental image:



Scratchwork:

$a+\frac{1}{n}<b \iff \frac{1}{n}<b-a \iff 1<n(b-a)$

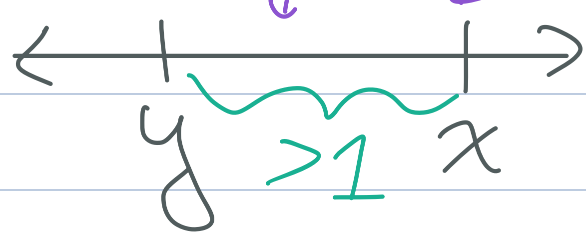
Pf: Let $y=b-a>0$. Since $1>0$, by Arch. Prop., $\exists\ n\in\mathbb{N}$ s.t. $n(b-a)=ny>1$. Thus $b-a>\frac{1}{n} \iff b>a+\frac{1}{n}$. $\square$

Lemma: If $x,y\in\mathbb{R}$ satisfy $1<x-y$, then $\exists\ m\in\mathbb{Z}$ s.t. $y<m<x$.

$y+1<x$

should be an $m\in\mathbb{Z}$ in here

Mental image:



Pf: By the first lemma, $\exists\ n\in\mathbb{N}$ s.t. $n>y$. Define $S=\{j\in\mathbb{Z}: y<j\leq n\}$. Then S is nonempty.
Furthermore, $\exists\ \tilde{n}\in\mathbb{N}$ s.t. $\tilde{n}>|y|$, so $-\tilde{n}<-|y|\leq y$.

Thus $|S| \le n + \tilde{n}$, that is, S can have at most $n+\tilde{n}$ elements, so it is a finite set. Thus, the minimum exists. Define $m := \min(S)$. By defn, $m \in \mathbb{Z}$, $y < m$. Likewise $m-1 \notin S$, so $m-1 \le y$. Therefore $y < m \le y+1 < x$. $\square$

Now, we apply the previous theorems to show

MAJOR THM #2

Thm ($\mathbb{Q}$ is dense in $\mathbb{R}$): If $a, b \in \mathbb{R}$ and $a < b$, $\exists r \in \mathbb{Q}$ s.t. $a < r < b$.

Mental image: $\mathbb{Q}$ is "sprinkled throughout" $\mathbb{R}$.

Pf: By the lemma, $\exists\, n \in \mathbb{N}$ s.t. $a + \frac{1}{n} < b$ $\Leftrightarrow 1 < bn - an$. By other lemma, $\exists\, m \in \mathbb{Z}$ s.t. $an < m < bn \Leftrightarrow a < \frac{m}{n} < b$. $\square$

We will use the symbols $+\infty$ and $-\infty$ to simplify our notation for suprema and infima.

Ex: For $a \in \mathbb{R}$

$$(a, +\infty) = \{x \in \mathbb{R} : a < x\}$$
$$= \{x \in \mathbb{R} : a < x < +\infty\}$$

Def: (Unbounded above/below)
For any nonempty set $S \subseteq \mathbb{R}$,
- if S is not bounded above, write $\sup(S) = +\infty$;

- if S is not bounded below, write $\inf(S) = -\infty$.

<u>Remark</u>: Given a nonempty set $S \subseteq \mathbb{R}$,

S has a supremum $\overset{\text{by defn of } \mathbb{R}}{\Longleftarrow}$ S is bounded above

$\Longrightarrow$

by defn of supremum

$\Longleftrightarrow \sup(S) \in \mathbb{R}$

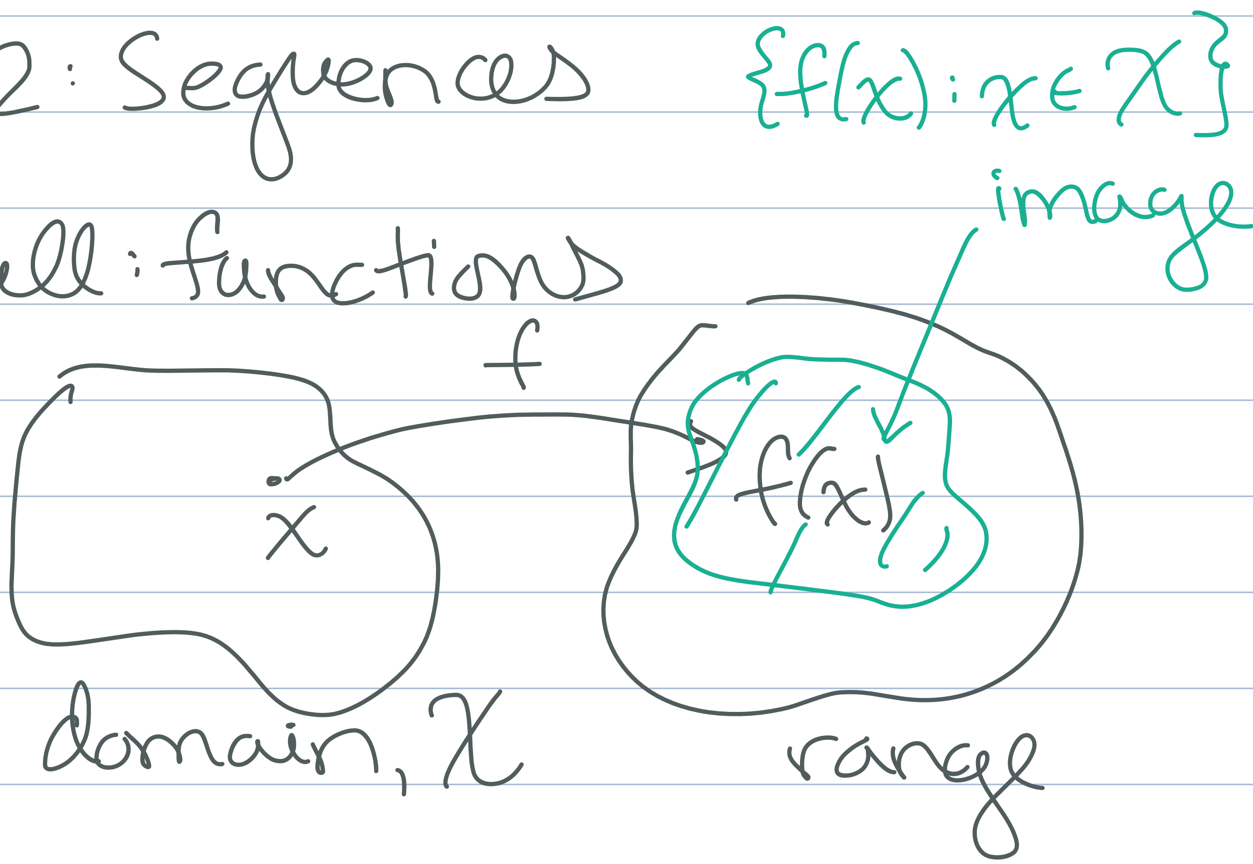
the supremum of S does not exist $\Longleftrightarrow$ S is not bounded above

$\Longleftrightarrow \sup(S) = +\infty$

This is convenient, since now for any nonempty $S \subseteq \mathbb{R}$, $\sup(S)$ has meaning.

Ch. 2: Sequences

Recall: functions



Def (sequence): A <u>sequence</u> is a function whose domain is a set of the form $\{m, m+1, m+2, \ldots\}$ for some $m \in \mathbb{Z}$. We will study sequences whose range is $\mathbb{R}$.

Typically, the domain will be either $\mathbb{N}$ or $\mathbb{N} \cup \{0\}$.

To emphacize that a sequence is a special type of function, instead of writing

$$f(n)$$

for its value at n, we write

$$s_n.$$

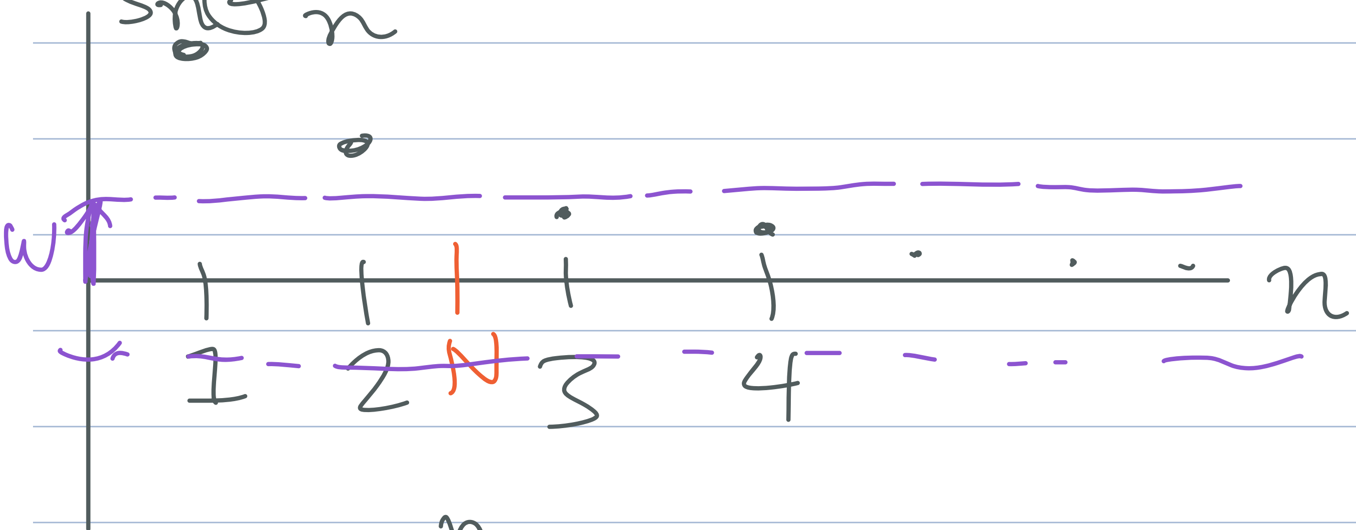
We will often specify a sequence by listing its values in order, ...

$$(s_1, s_2, s_3, \ldots, s_n, \ldots).$$

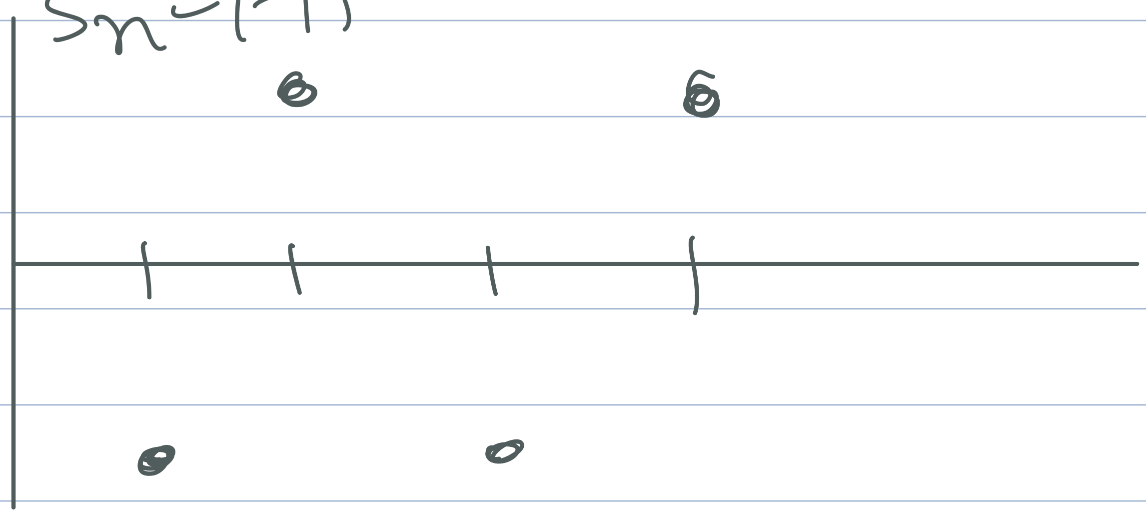
Ex: If $s_n = \frac{1}{n}$, $n \in \mathbb{N}$, the sequence is $\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots\right)$

Heuristically, a sequences "converges" to some $s \in \mathbb{R}$ if the values of s_n "get close and stay close" to s, for n sufficiently large.

$s_n = \frac{1}{n}$



$s_n = (-1)^n$



Def (convergence): A sequence s_n <u>converges</u> $s \in \mathbb{R}$ provided that $\forall \varepsilon > 0$, $\exists N \in \mathbb{R}$ so that $n > N$ ensures $|s_n - s| < \varepsilon$.

We call $s \in \mathbb{R}$ the <u>limit</u> of s_n and write $\lim_{n \to \infty} s_n = s$ or $s_n \to s$.

A sequence that does not converge to any $s \in \mathbb{R}$ is said to <u>diverge</u>.