Lecture 4

CS 117, S25

Announcements:

- No class next week (Tues Apr 15, Thur Apr 17)

Recall:

MAJOR THM #1

Thm (Archimedean Property): If $a, b \in \mathbb{R}$ satisfy $a > 0$, $b > 0$, then $\exists\, n \in \mathbb{N}$ s.t. $na > b$.

Lemma: For any $x \in \mathbb{R}$, $\exists\, n \in \mathbb{N}$ s.t. $x < n$.

Lemma: For $a, b \in \mathbb{R}$, $a < b$, $\exists\, n \in \mathbb{N}$ s.t. $a + \frac{1}{n} < b$.

Lemma: If $x, y \in \mathbb{R}$ satisfy $1 < x - y$, then $\exists\ m \in \mathbb{Z}$ s.t. $y < m < x$.

MAJOR THM #2

Thm ($\mathbb{Q}$ is dense in $\mathbb{R}$): If $a, b \in \mathbb{R}$ and $a < b$, $\exists\ r \in \mathbb{Q}$ s.t. $a < r < b$.

Def (Unbounded above/below) For any nonempty set $S \subseteq \mathbb{R}$,

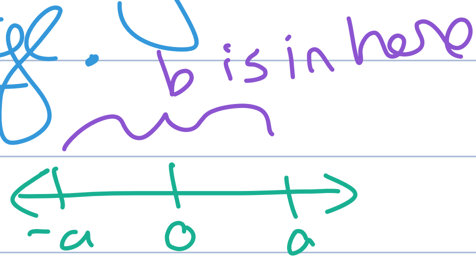
- if S is not bounded above, write $\sup(S) = +\infty$;
- if S is not bounded below, write $\inf(S) = -\infty$.

Def (sequence): A sequence is a function whose domain is a set of the form $\{m, m+1, m+2, \dots\}$ for some $m \in \mathbb{Z}$. We will study sequences whose range is $\mathbb{R}$.

<u>Def</u> (convergence): A sequence s_n <u>converges</u> $s \in \mathbb{R}$ provided that $\forall \varepsilon > 0$, $\exists N \in \mathbb{R}$ so that $n > N$ ensures $|s_n - s| < \varepsilon$.

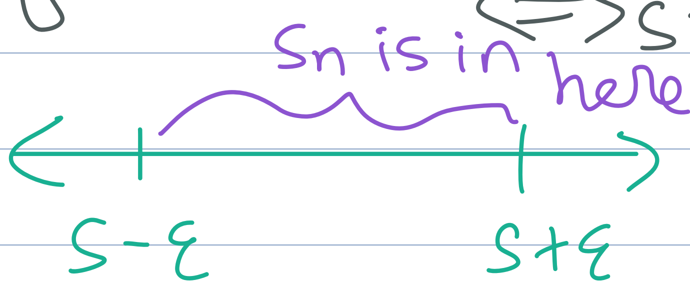
We call $s \in \mathbb{R}$ the <u>limit</u> of s_n and write $\lim_{n \to \infty} s_n = s$ or $s_n \to s$.

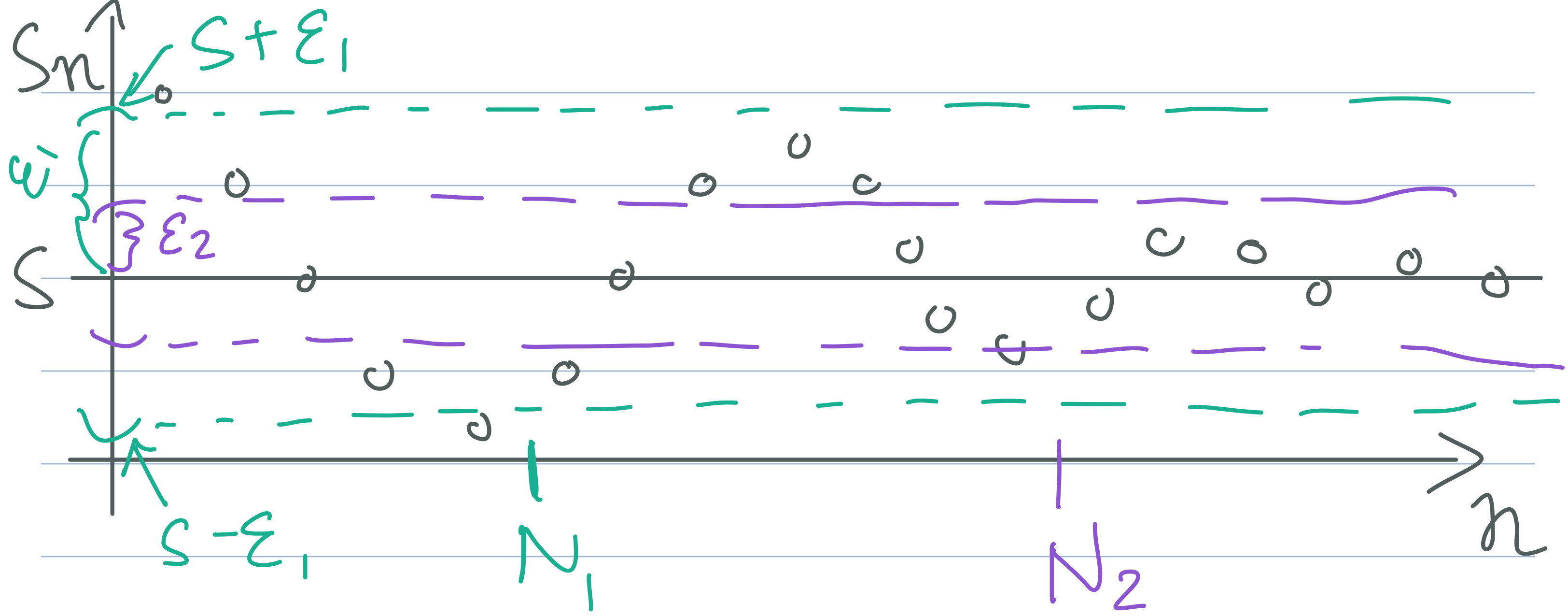
A sequence that does not converge to any $s \in \mathbb{R}$ is said to <u>diverge</u>.



<u>Remark</u>:

- Recall: $|b| < a \iff -a < b < a$
- Similarly, $|s_n - s| < \varepsilon \iff -\varepsilon < s_n - s < \varepsilon$
  $\iff s - \varepsilon < s_n < s + \varepsilon$





Ex: Consider the sequence $s_n = \frac{1}{n^2}$.
We expect that $\lim_{n\to\infty} \frac{1}{n^2} = 0$.
Let's prove it!

Scratchwork:
$\left|\frac{1}{n^2} - 0\right| < \varepsilon \iff \frac{1}{n^2} < \varepsilon \iff \underbrace{\frac{1}{\sqrt{\varepsilon}}}_{N} < n.$

Proof: Fix arbitrary $\varepsilon > 0$. Let $N = \frac{1}{\sqrt{\varepsilon}}$.
Then $n > N$ ensures
$n > \frac{1}{\sqrt{\varepsilon}} \iff \frac{1}{n^2} < \varepsilon \iff \left|\frac{1}{n^2} - 0\right| < \varepsilon$.
Thus, $\lim_{n\to\infty} \frac{1}{n^2} = 0$.

also true for $N = \frac{1}{\sqrt{\varepsilon}} + \pi$

Ex: Consider the $s_n = (-1)^n$. We expect it $\underbrace{\text{does not converge}}_{\iff \text{diverges}}$. Let's prove it!

Pf: We proceed by contradiction. Assume, for the sake of contradiction that s_n converges to s. By defn of convergence, $\forall \varepsilon > 0$, $\exists N$ s.t. $n > N$ ensures $|s_n - s| < \varepsilon$. Let $\varepsilon = 1$. Then $\exists N$ s.t. $n > N$ ensures $|s_n - s| < 1 \iff s - 1 < s_n < s + 1$.

Claim: there exist odd an even n satisfying $n > N$.

For n odd, $s_n = -1$, so $s - 1 < s_n$ implies $s < 0$. For n even, $s_n = 1$, so $s_n < s + 1$ implies $s > 0$.

There is no such $s \in \mathbb{R}$.
This is a contradiction.

Thus s_n does not converge to any $s \in \mathbb{R}$; hence it diverges. $\square$

<u>Ex</u>: Consider sequence $s_n = \dfrac{2n-1}{3n+2} = \dfrac{2-\frac{1}{n}}{3+\frac{2}{n}}$.

What is the limit?

Scratchwork:

$$\left| \frac{2n-1}{3n+2} - \frac{2}{3} \right| < \varepsilon$$

$$\Leftrightarrow \left| \frac{3\cdot(2n-1) - 2(3n+2)}{3(3n+2)} \right| < \varepsilon$$

$$\Leftrightarrow \left| \frac{-7}{3(3n+2)} \right| < \varepsilon$$

$$\Leftrightarrow \frac{7}{3(3n+2)} < \varepsilon \Leftarrow \frac{7}{3(3n)} < \varepsilon \Leftrightarrow \overbrace{\frac{7}{9\varepsilon}}^{N} < n$$

Pf: Fix arbitrary $\varepsilon > 0$. Let $N = \frac{7}{9\varepsilon}$.
Then, $n > N$ ensures

$$\frac{7}{9\varepsilon} < n \iff \frac{7}{3(3n)} < \varepsilon \implies \frac{7}{3(3n+2)} < \varepsilon$$

$$\iff \left| \frac{2n-1}{3n+2} - \frac{2}{3} \right| < \varepsilon.$$

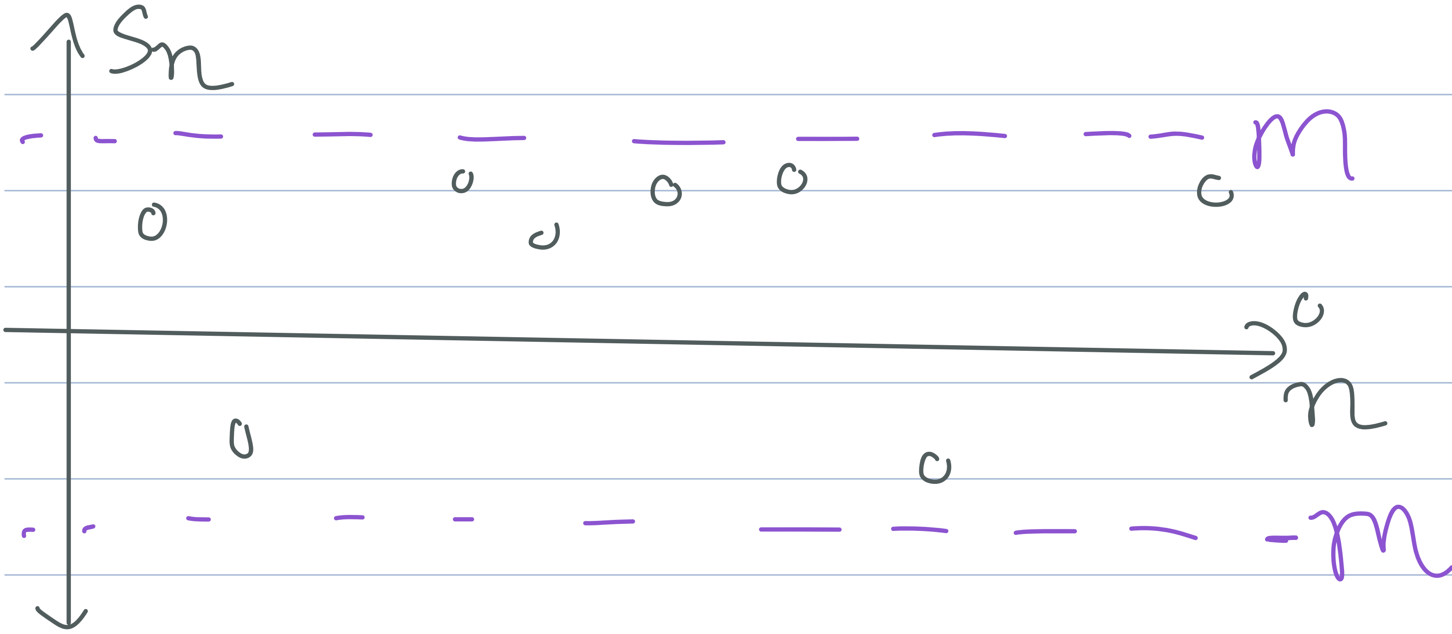
Therefore $\lim_{n\to\infty} \frac{2n-1}{3n+2} = \frac{2}{3}$. □

Another type of sequence is...

Def: A sequence s_n is <u>bounded</u> if there exists $M \in \mathbb{R}$ s.t. $|s_n| \leq M$ for all n.

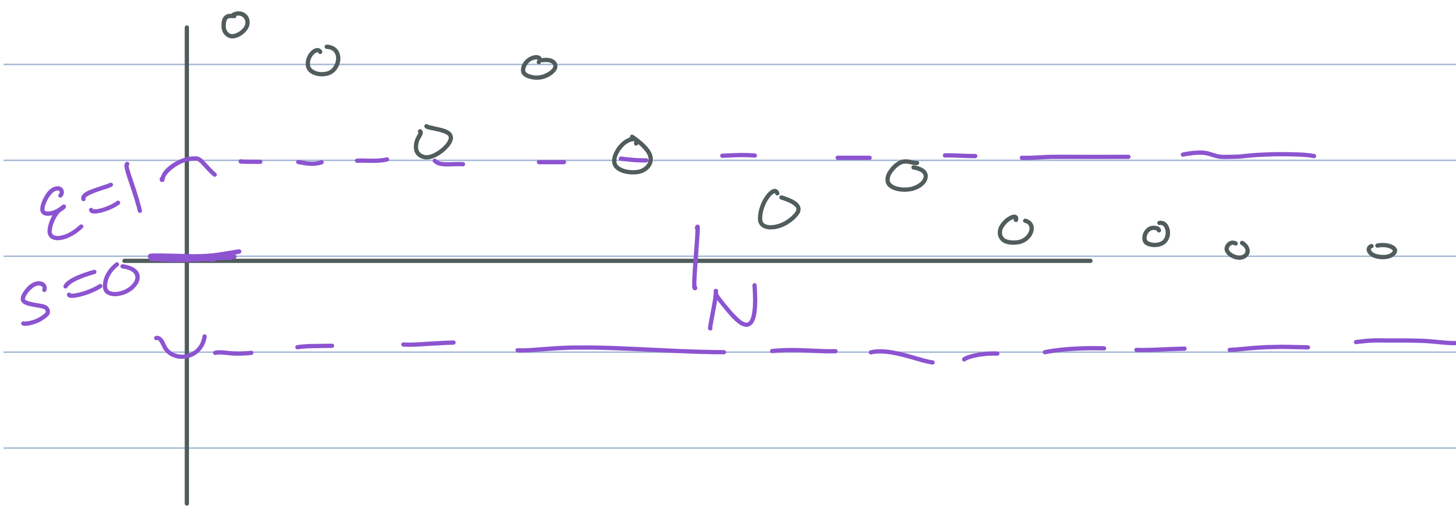
$$\Updownarrow$$

$$-M \leq s_n \leq M$$



Rmk: A sequence is bounded iff the set $S=\{s_n : n\in\mathbb{N}\}\subseteq\mathbb{R}$ is bounded. (HW 3)

Thm: Convergent sequences are bounded.



Pf: Suppose s_n converges to $s \in \mathbb{R}$. Then, for $\varepsilon = 1$, $\exists\, N$ s.t. $n > N$ ensures $|s_n - s| < 1$.

By reverse triangle inequality, $|s_n| - |s| \le \big||s_n| - |s|\big| \le |s_n - s| < 1$, so $|s_n| < \underbrace{|s|+1}$ for $n > N$.

idea: M will relate to this.

Interlude: Given $a \in \mathbb{R}$,
$\lceil a \rceil = \min\{z \in \mathbb{Z} : z \ge a\}$ "ceiling"
$\lfloor a \rfloor = \max\{z \in \mathbb{Z} : z \le a\}$ "floor"

Let $S := \{|s_1|, |s_2|, \dots, |s_{\lfloor N \rfloor}|\} = \{|s_k| : k \in \mathbb{N}, k \le N\}$

Define, $M_S := \max S$, $M = \max\{M_S, |s|+1\}$

By defn, $|s_n| \le M \;\; \forall\, n \in \mathbb{N}$. $\square$

Rmk: Not all bounded sequences are convergent, $s_n = (-1)^n$.

$\square$