Lecture 5

CS 117, S25

Announcements:

- Makeup lecture, this Friday, Apr 25, 11-12:15
- Midterm 1 in one week, on Tues, Apr 29
- No office hours on Mon, Apr 28

Recall:

Def: A sequence s_n is bounded if there exists $M \in \mathbb{R}$ s.t. $|s_n| \leq M$ for all n.

Rmk: A sequence is bounded iff the set $S = \{s_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$ is bounded. (HW 3) $s_n = (-1)^n$, $S = \{-1, 1\}$

Thm: (real valued) Convergent sequences are bounded.

Rmk: Not all bounded sequences are convergent (e.g. $(-1)^n$)

Now, we will prove limit theorems, which will enable you to study convergence behavior of complicated sequences in terms of components.

Thm (limit of sum is sum limits): If s_n and t_n are convergent sequences, then $\lim_{n\to\infty}(s_n+t_n) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n$.

Ex: $\lim_{n\to\infty} \frac{\pi}{n} + \frac{\sqrt{2}}{n^2} = \lim_{n\to\infty} \frac{\pi}{n} + \lim_{n\to\infty} \frac{\sqrt{2}}{n^2} = 0+0=0$

Pf: Let $s := \lim_{n\to\infty} s_n$, $t := \lim_{n\to\infty} t_n$.

Note that

$$\underbrace{|(s_n+t_n)-(s+t)|}_{(s_n-s)+(t_n-t)} \overset{\Delta \text{ ineq}}{\leq} \underbrace{|s_n-s|}_{<\varepsilon/2} + \underbrace{|t_n-t|}_{\varepsilon/2}$$

want: $<\varepsilon/2$ $\quad \varepsilon/2$

Fix $\varepsilon > 0$ arbitrary. Since s_n converges to s, t_n converges to t, and $\frac{\varepsilon}{2} > 0$ $\exists\, N_s, N_t \in \mathbb{R}$ s.t.

$$n > N_s \implies |s_n - s| < \frac{\varepsilon}{2}$$
$$n > N_t \implies |t_n - t| < \frac{\varepsilon}{2}.$$

Then $n > \max\{N_s, N_t\}$ ensures

$$|(s_n + t_n) - (s+t)| \leq |s_n - s| + |t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore $\lim_{n\to\infty} s_n + t_n = s + t$. $\square$

<u>Rmk</u>: Consider $s_n = (-1)^n$, $t_n = (-1)^{n+1}$.

Then $0 = \lim_{n\to\infty} s_n + t_n \neq \underbrace{\lim_{n\to\infty} s_n}_{} + \underbrace{\lim_{n\to\infty} t_n}_{}$

$\rightarrow$ do not exist

The hypothesis "convergent sequences" is necessary.

Thm (limit of product is product of limits): If s_n and t_n are convergent sequences,
$$\lim_{n\to\infty} s_n t_n = \left(\lim_{n\to\infty} s_n\right)\left(\lim_{n\to\infty} t_n\right).$$

Pf: Let $s := \lim_{n\to\infty} s_n$, $t := \lim_{n\to\infty} t_n$.

Note that add and subtract
$$\begin{aligned} |s_n t_n - st| &= |s_n t_n - s_n t + s_n t - st| \\ &= |s_n(t_n - t) + t(s_n - s)| \\ &\overset{\Delta\text{ineq}}{\le} |s_n(t_n - t)| + |t(s_n - s)| \\ &= |s_n||t_n - t| + |t||s_n - s| \end{aligned}$$

Since all convergent sequences are bounded, $\exists M > 0$ s.t. $|s_n| \le M$ $\forall n \in \mathbb{N}$. Thus,

$$|s_n t_n - st| \le M|t_n - t| + |t||s_n - s|$$

Fix $\varepsilon > 0$. Since t_n converges to t and s_n converges to s,

$\exists\ N_t, N_s$ s.t.

$n > N_t$ ensures $|t_n - t| < \frac{\varepsilon}{2m}$

$n > N_s$ ensures $|s_n - s| < \begin{cases} \frac{\varepsilon}{2|t|} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$

Therefore, $n > \max\{N_t, N_s\}$ ensures

$$|s_n t_n - st| \leq m\,|t_n - t| + |t|\,|s_n - s| \qquad \square$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Remark on defn of convergence:
A sequence s_n converges to a limit s if for all $\varepsilon > 0$,

$\begin{cases} \exists\, N \in \mathbb{R} \\ \exists\, N \in \mathbb{N} \end{cases}$

such that

$\begin{cases} n > N \\ n \geq N \end{cases}$

ensures

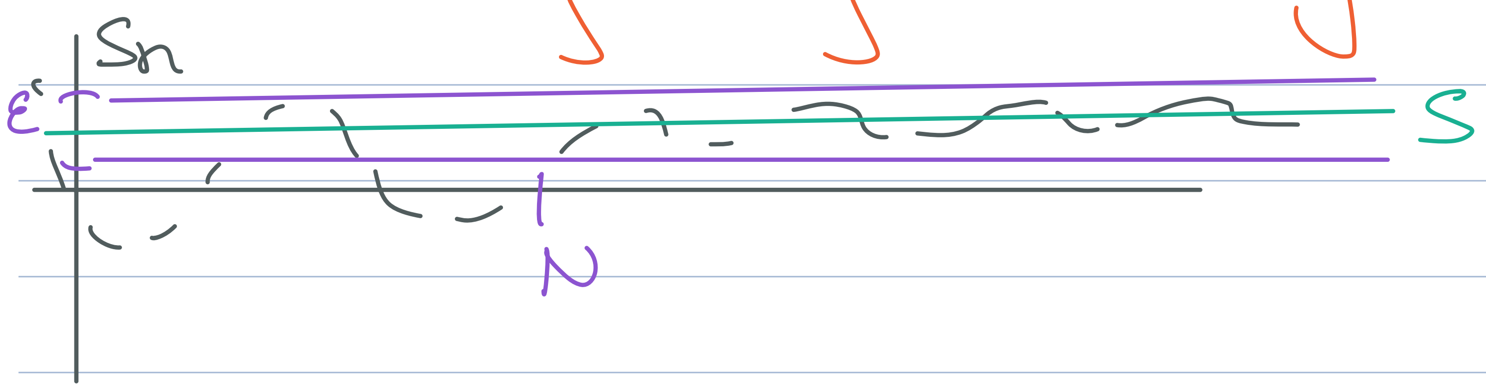
$\begin{cases} |s_n - s| < \varepsilon \\ |s_n - s| \leq \varepsilon \end{cases}$

any combination of choices leads to an equivalent defn :)

Thm (limit of quotient is quotient of limits): If s_n and t_n are convergent sequences and $\lim_{n\to\infty} s_n \neq 0$

$$\lim_{n\to\infty}\left(\frac{t_n}{s_n}\right) = \frac{\lim_{n\to\infty} t_n}{\lim_{n\to\infty} s_n}$$

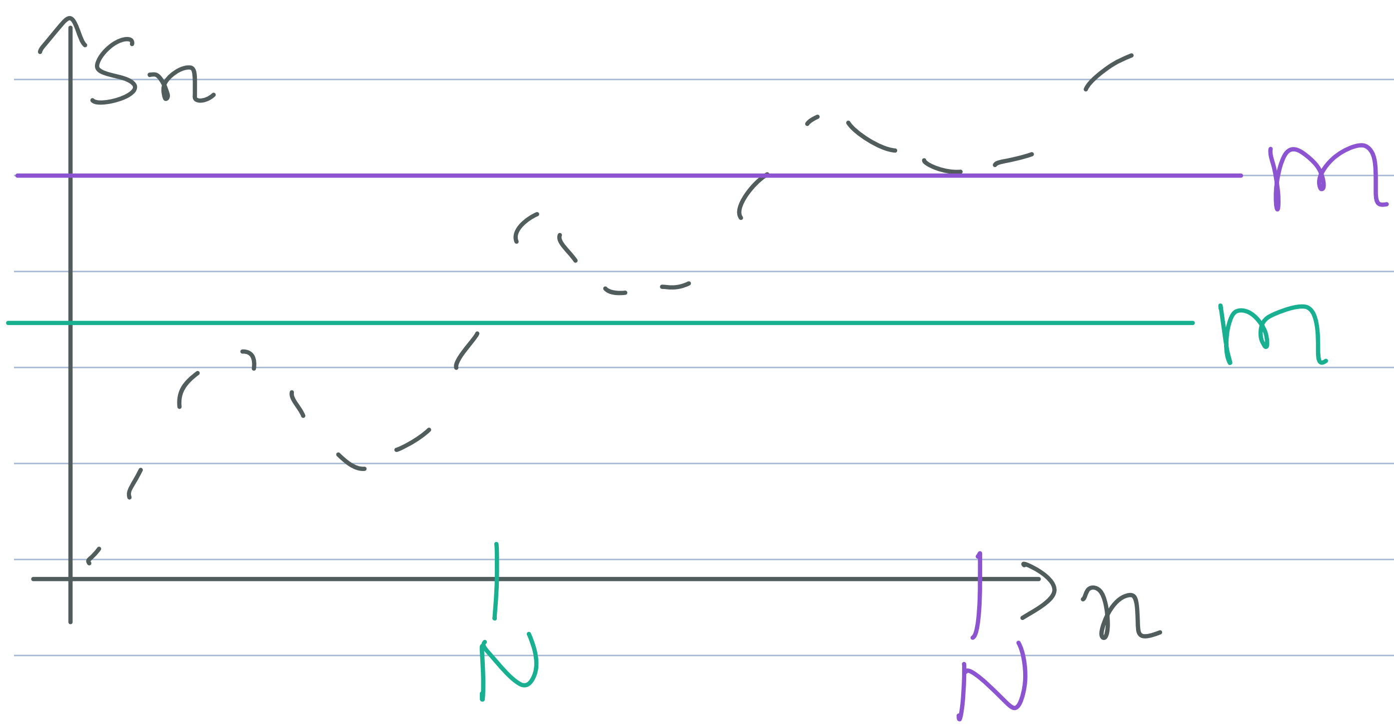
this ensures that $s_n = 0$ for at most finitely many values of $n \in \mathbb{N}$.



Ex: What is the limit of $s_n = n^2$?

Def (diverges to $+\infty$ or $-\infty$): A sequence s_n <u>diverges to $+\infty$</u> if $\forall M>0$, $\exists N\in\mathbb{R}$ so that $n>N$ ensures $s_n>M$.
We write $\lim_{n\to\infty} s_n = +\infty$.

OTOH, s_n diverges to $-\infty$ if $\forall M<0$, $\exists N\in\mathbb{R}$ so that $n>N$ ensures $s_n<M$.
We write $\lim_{n\to\infty} s_n = -\infty$.



Recall: All convergent sequences are bounded. Thus, it's clear that any sequence that diverges to either $+\infty$ or $-\infty$ does not "converge", hence diverges.

Remark: We will say s_n "has a limit" or "the limit exists" if either:

① s_n converges $\quad \lim_{n\to\infty} s_n \in \mathbb{R}$

② s_n diverges to $\pm\infty$ $\quad \lim_{n\to\infty} s_n \in \{+\infty, -\infty\}$

Some more limit theorems...

Thm: Suppose $\lim_{n\to\infty} s_n = +\infty$ and $\lim_{n\to\infty} t_n > 0$. Then $\lim_{n\to\infty} s_n t_n = +\infty$.

Case 1: t_n converges to $t > 0$

Case 2: t diverges to $+\infty$

Pf: HW

<u>Thm</u>: Suppose s_n is a sequence of positive numbers. Then $\lim_{n\to\infty} s_n = +\infty \iff \lim_{n\to\infty} \frac{1}{s_n} = 0$.

<u>Pf</u>: First, show $\lim_{n\to\infty} s_n = +\infty \Rightarrow \lim_{n\to\infty} \frac{1}{s_n} = 0$.

Fix $\varepsilon > 0$. Note that

$$\left|\frac{1}{s_n}\right| < \varepsilon \overset{s_n \text{ positive}}{\iff} \frac{1}{s_n} < \varepsilon \iff \frac{1}{\varepsilon} < s_n$$

Since $\overbrace{\frac{1}{\varepsilon}}^{m} > 0$, by definition of divergence to $+\infty$, $\exists N$ s.t. $n > N$ ensures $s_n > \frac{1}{\varepsilon}$, hence $\left|\frac{1}{s_n} - 0\right| < \varepsilon$.

This shows $\lim_{n\to\infty} \frac{1}{s_n} = 0$.

To see the converse,
fix $m>0$ arbitrary.
Since $\frac{1}{m}>0$, the convergence
of $\frac{1}{s_n}$ to 0 implies $\exists$ N s.t
$n>N$ ensures

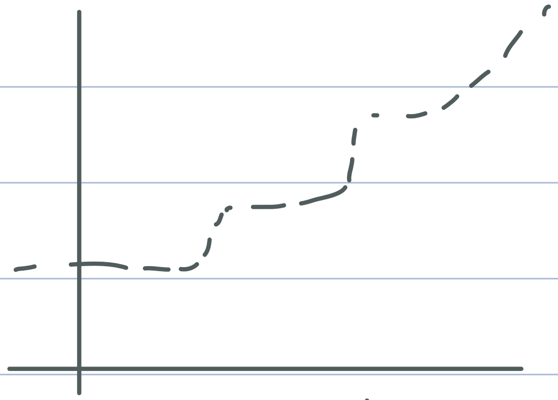
$$\left|\frac{1}{s_n}-0\right|<\frac{1}{m} \iff \frac{1}{s_n}<\frac{1}{m} \iff m<s_n.$$

Therefore $\lim_{n\to\infty} s_n=+\infty$. $\square$

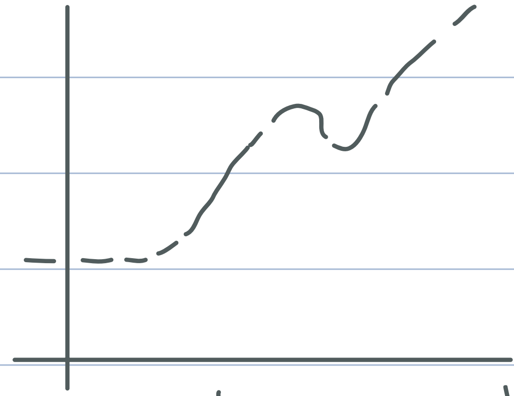
== End of Material for Midterm 1 ==

Another important class of sequence...

Def:
A sequence s_n is increasing if $s_n \le s_{n+1}\ \forall n$
decreasing if $s_n \ge s_{n+1}\ \forall n$
monotone if either increasing or decreasing.



monotone ✓



not monotone ✗

Remark: If s_n is increasing, then $s_n \le s_m$ whenever $n \le m$.

MAJOR THM # 3

Thm: All bounded monotone sequence converge.

Mental picture: