Lecture 6

CS 117, S25

Announcements:

- Makeup lecture, this Friday, Apr 25, 11-12:15
- Midterm 1 in one week, on Tues, Apr 29
- No office hours on Mon, Apr 28
- DSP

Recall:

Thm (limit of sum is sum of limits): If s_n and t_n are convergent sequences, then $\lim_{n\to\infty}(s_n+t_n) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n$.

Things get weird if $\lim_{n\to\infty} s_n = +\infty$, $\lim_{n\to\infty} t_n = -\infty$.

- $s_n = n$, $t_n = -n$, $s_n + t_n \xrightarrow{n\to\infty} 0$
- $s_n = n^2$, $t_n = -n$, $s_n + t_n \to +\infty$
- $s_n = n$, $t_n = -n+\pi$, $s_n + t_n \to \pi$

Thm (limit of product is product of limits): If s_n and t_n are convergent sequences, $\lim_{n\to\infty} s_n t_n = (\lim_{n\to\infty} s_n)(\lim_{n\to\infty} t_n)$.

Things get weird if $s_n \to +\infty$, $t_n \to 0$

- $s_n = n$, $t_n = \frac{1}{n}$, $s_n t_n \to 1$
- $s_n = n^2$, $t_n = \frac{1}{n}$, $s_n t_n \to +\infty$

Thm (limit of quotient is quotient of limits): If s_n and t_n are convergent sequences and $\lim_{n\to\infty} s_n \neq 0$

$$\lim_{n\to\infty}\left(\frac{t_n}{s_n}\right) = \frac{\lim_{n\to\infty} t_n}{\lim_{n\to\infty} s_n}.$$

Def (diverges to $+\infty$ or $-\infty$): A sequence s_n diverges to $+\infty$ if $\forall M > 0$, $\exists N \in \mathbb{R}$ so that $n > N$ ensures $s_n > M$.
We write $\lim_{n\to\infty} s_n = +\infty$.

OTOH, s_n diverges to $-\infty$ if $\forall M < 0$, $\exists N \in \mathbb{R}$ so that $n > N$ ensures $s_n < M$.
We write $\lim_{n\to\infty} s_n = -\infty$.

Remark: We will say s_n "has a limit" or "the limit exists" if either:

① s_n converges $\lim_{n\to\infty} s_n \in \mathbb{R}$

② s_n diverges to $\pm\infty$ $\lim_{n\to\infty} s_n \in \{+\infty, -\infty\}$

Thm: Suppose $\lim_{n\to\infty} s_n = +\infty$ and $\lim_{n\to\infty} t_n > 0$. Then $\lim_{n\to\infty} s_n t_n = +\infty$.

Thm: Suppose s_n is a sequence of positive numbers. Then
$$\lim_{n\to\infty} s_n = +\infty \iff \lim_{n\to\infty} \frac{1}{s_n} = 0.$$

End of Material for Midterm 1

A related fact you will prove on HW4...
$$\lim_{n\to\infty} s_n = +\infty \iff \lim_{n\to\infty} -s_n = -\infty.$$

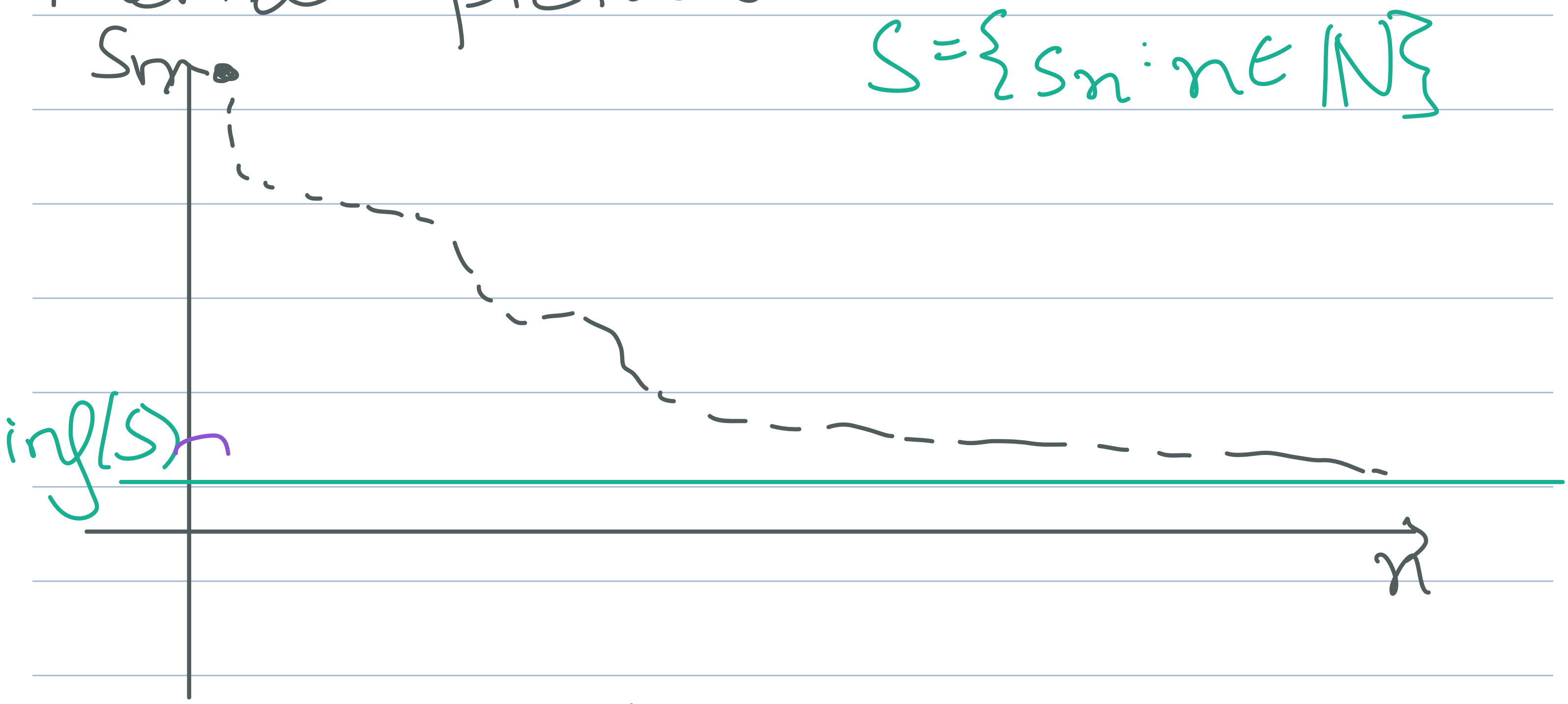
Def:
A sequence s_n is increasing if $s_n \le s_{n+1}\ \forall n$
decreasing if $s_n \ge s_{n+1}\ \forall n$
monotone if either increasing or decreasing.

Remark: If s_n is increasing, then $s_n \le s_m$ whenever $n \le m$.

MAJOR THM # 3

Thm: All bounded monotone sequence converge.

Mental picture:



Pf: $\boxed{\text{Case 1}}$ Suppose s_n is a bounded increasing sequence. Define $S := \{s_n : n \in \mathbb{N}\}$, which is a bounded set. We will show $\lim_{n\to\infty} s_n = \sup(S)$.

Fix $\varepsilon > 0$. We have $s_n \leq \sup(S) < \sup(S)+\varepsilon$ for all $n \in \mathbb{N}$. Since $\sup(S)$ is the least upper bound of S, $\sup(S)-\varepsilon$ is not an upper bound of S. Thus, $\exists\, n_0 \in \mathbb{N}$ s.t. $s_{n_0} > \sup(S)-\varepsilon$. Therefore, $\forall\, n > n_0$, we have

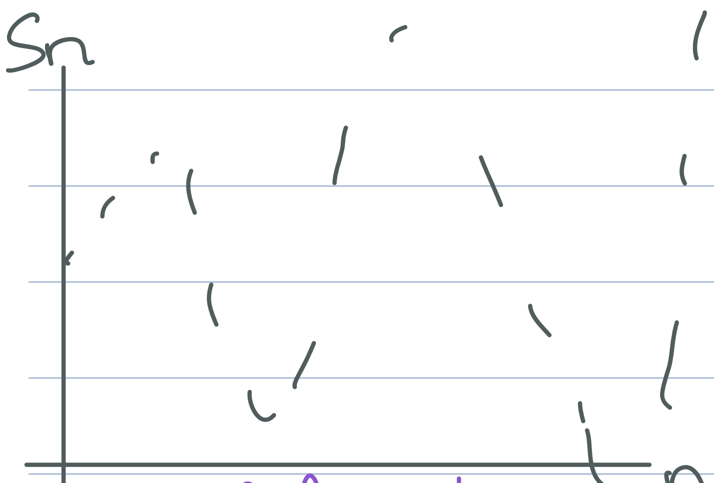
$$\sup(S)-\varepsilon < s_n < \sup(S)+\varepsilon \iff |s_n - \sup(S)| < \varepsilon.$$

Thus, $\lim_{n\to\infty} s_n = \sup(S)$.

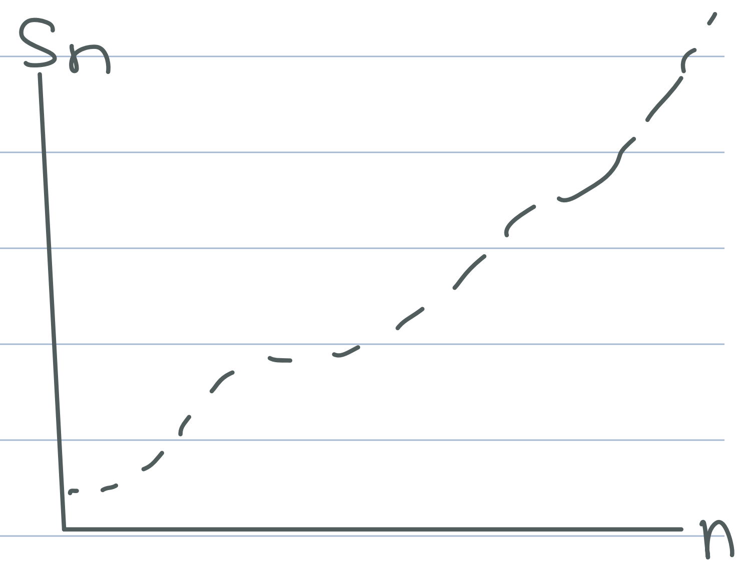
$\boxed{\text{Case 2}}$ Suppose s_n is a bounded decreasing sequence. Let $t_n := -s_n$. Then t_n is a bounded increasing sequence so, by Case 1, it converges.

Since $s_n = -t_n = (-1)(t_n)$ and the sequences $(-1,-1,-1,-1,\ldots)$ and t_n converge, since the limit of the product is the product of the limits, s_n converges. $\square$

In fact, even <u>unbounded</u> monotone sequences have a limit.



unbounded, not monotone



bounded: $\exists\, M$ s.t. $|s_n| \leq M \;\forall n$

unbounded: $\forall M, \exists n \in \mathbb{N}$ s.t. $|s_n| > M$

<u>Thm</u>: If s_n is an unbounded increasing sequence, $\lim_{n\to\infty} s_n = +\infty$. If s_n is an unbounded decreasing sequence, $\lim_{n\to\infty} s_n = -\infty$.

<u>Pf</u>: $\boxed{\text{Case 1}}$ Suppose s_n is unbounded, increasing. Fix $M>0$. Since s_n is increasing, it is bounded below ($s_n \geq s_1$ $\forall n \in \mathbb{N}$).
Thus, the sequence cannot be bounded above (otherwise, $\exists \tilde{M}$ s.t. $s_1 \leq s_n \leq \tilde{M} \Rightarrow \exists \tilde{\tilde{M}}$ s.t. $|s_n| \leq \tilde{\tilde{M}}$).
Hence M is not an upper bound for the sequence, so $\exists n_0 \in \mathbb{N}$ s.t. $s_{n_0} > M$. Since s_n is increasing, $\forall n > n_0$ $s_n > M$.
Thus, $\lim_{n\to\infty} s_n = +\infty$.

$\boxed{\text{Case 2}}$ Suppose s_n is unbounded and decreasing. Then $t_n := -s_n$, is unbounded and increasing.

By Case 1, $\lim_{n\to\infty} t_n = +\infty$.

$\{s_n \geq s_{n+1} \overset{?}{\iff} -s_n \leq -s_{n+1}\}$

By fact, $\lim_{n\to\infty} s_n = \lim_{n\to\infty} -t_n = -\infty$. $\square$

unbounded: $\forall M>0, \exists n\in\mathbb{N}$ s.t. $|s_n|>M$

unbdd + increasing $\iff$

In summary, if s_n is monotone:

$$\lim_{n\to\infty} s_n = \begin{cases} +\infty & \text{if } s_n \text{ unbdd above} \\ s, \text{ for } s\in\mathbb{R} & \text{if } s_n \text{ is bdd} \\ -\infty & \text{if } s_n \text{ unbdd below} \end{cases}$$

$\Updownarrow$

unbdd + decreasing

Thus, the limit of any monotone sequence always exists.

In general, we can reduce the study of arbitrary sequences to components that are monotone.

Even for arbitrary sequences, there is a generalization of the notion of limit that always exist

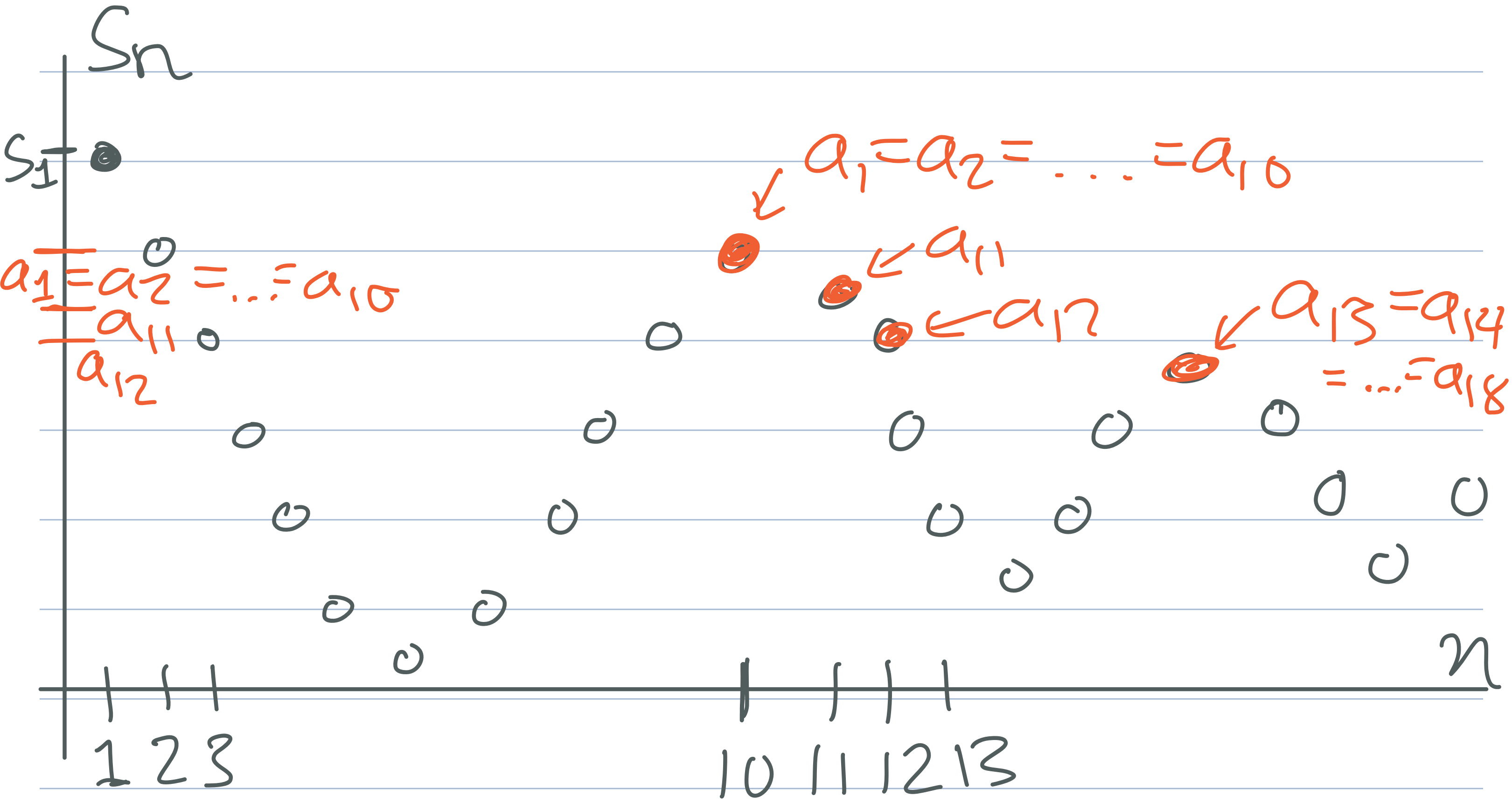
sup generalizes max

Def (limsup/liminf) For any sequence s_n,

$$\limsup_{n\to\infty} s_n = \lim_{N\to\infty} \overbrace{\sup\{s_n : n > N\}}^{a_N}$$

Why does $\limsup_{n\to\infty} s_n$ always exist?

- Since $\{s_n : n > \underline{N}\} \supseteq \{s_n : n > N+1\}$, $\sup(\{s_n : n > N\}) \geq \sup(\{s_n : n > N+1\})$, $a_N \geq a_{N+1}$, so a_N is decreasing



$a_1 = \sup\{s_n : n>1\}$
$a_2 = \sup\{s_n : n>2\}$

Warning: it is possible that $a_N = +\infty$.

However, the only possibilities are...

in fact, N_0 can always be chosen to be zero!

① $a_N = +\infty$ for $N \leq N_0$, $a_N \in \mathbb{R}$ for $N > N_0$
$\Rightarrow$ we use usual defn for limit of real sequence.

② $a_N = +\infty$ for all $N \Rightarrow$ we say $\lim\limits_{N\to\infty} a_N = +\infty$