Lecture 7

CS 117, S25

Announcements:

- Midterm 1 on Tues
- No office hours on Mon, Apr 28

Recall:

MAJOR THM # 3

Thm: All bounded monotone sequence converge.

Thm: If s_n is an unbounded increasing sequence, $\lim_{n\to\infty} s_n = +\infty$.
If s_n is an unbounded decreasing sequence, $\lim_{n\to\infty} s_n = -\infty$.

In summary, if s_n is monotone:

$$\lim_{n\to\infty} s_n = \begin{cases} +\infty & \text{if } s_n \text{ unbdd above} \\ s, \text{ for } s \in \mathbb{R} & \text{if } s_n \text{ is bdd} \\ -\infty & \text{if } s_n \text{ unbdd below} \end{cases}$$

Thus, the limit of any monotone sequence always exists!

<u>Def</u> (limsup/liminf) For any sequence s_n,

$$\limsup_{n\to\infty} s_n = \lim_{N\to\infty} \overbrace{\sup\{s_n : n > N\}}^{a_N}$$

$$\liminf_{n\to\infty} s_n = \lim_{N\to\infty} \underbrace{\inf\{s_n : n > N\}}_{b_N}$$

Remark:

- a_N is decreasing
- either $a_N = +\infty$ $\forall N$ or a_N is real valued
- b_N is increasing
- either $b_N = -\infty$ $\forall N$ or b_N is real valued

Examples

$$\limsup_{n\to\infty} (-1)^n = \lim_{N\to\infty} \overbrace{\sup\{(-1)^n : n > N\}}^{=1 \text{ for all } N} = 1$$

$$\liminf_{n\to\infty} (-1)^n = \lim_{N\to\infty} \underbrace{\inf\{(-1)^n : n > N\}}_{=-1 \text{ for all } N} = -1$$

$$\limsup_{n\to\infty} \frac{1}{n^2} = \lim_{N\to\infty} \overbrace{\sup\{\tfrac{1}{n^2} : n > N\}}^{\frac{1}{(N+1)^2}} = 0$$

$$\liminf_{n\to\infty} \frac{1}{n^2} = \lim_{N\to\infty} \underbrace{\inf\{\tfrac{1}{n^2} : n > N\}}_{=0 \text{ for all } N} = 0$$

Thm: Given a sequence s_n,

MAJOR THM #4

$\lim_{n\to\infty} s_n$ exists $\iff \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$.

Furthermore, if either of these equivalent conditions holds,

$$\lim_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n.$$

Remarks (that we will use in proof):

① If $s < M \ \forall s \in S$, then $\sup(S) \le M$.

② $b_N = \inf\{s_n : n > N\} \le s_n \le \sup\{s_n : n > N\} = a_N$

for any $n > N$

③ Fact: If r_n and t_n are sequences whose limits exist and that satisfy $r_n \le t_n \ \forall n \in \mathbb{N}$, then $\lim_{n\to\infty} r_n \le \lim_{n\to\infty} t_n$.

Pf: HW4

Immediate consequence:

$$\liminf_{n\to\infty} s_n = \lim_{N\to\infty} b_N \le \lim_{N\to\infty} a_N = \limsup_{n\to\infty} s_n$$

(4) Fact: If either limit exists, then

$$\overline{\lim_{n\to\infty}} -t_n = -\lim_{n\to\infty} t_n.$$

<u>Pf</u>: HW4

Consequence:

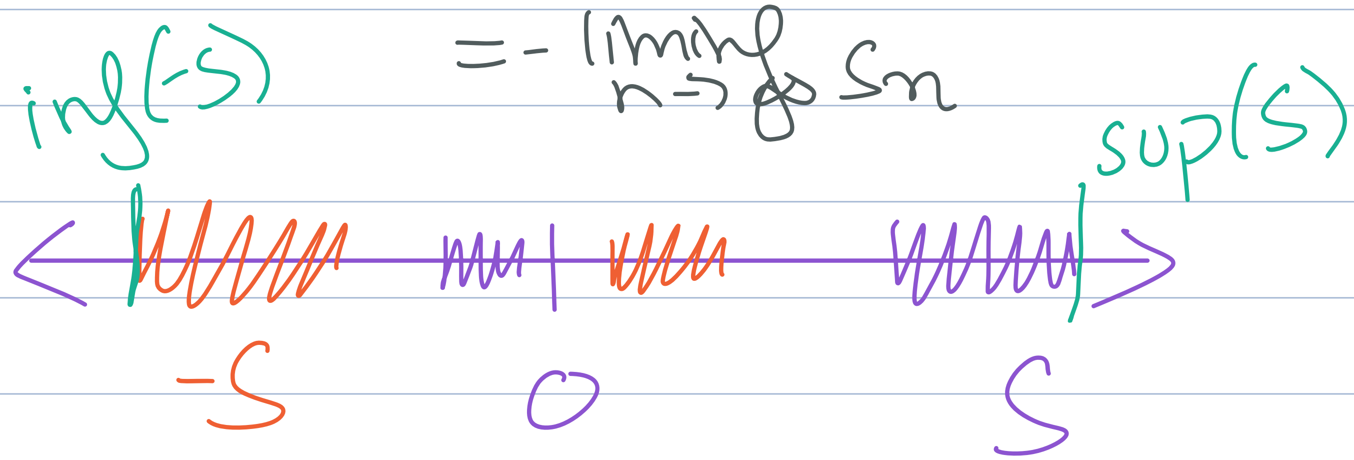
$$\limsup_{n\to\infty} (-s_n) = \lim_{N\to\infty} \sup\{-s_n : n>N\}$$

$$= \lim_{N\to\infty} -\inf\{s_n : n>N\}$$

Fact ↓

$$= -\lim_{N\to\infty} \inf\{s_n : n>N\}$$

$$= -\liminf_{n\to\infty} s_n$$



Pf:

$\boxed{\text{Suppose } \lim_{n\to\infty} s_n \text{ exists. WTS } \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = \lim_{n\to\infty} s_n}$

$\boxed{\text{Case 1}}$ $\lim_{n\to\infty} s_n = -\infty$. WTS a_N and b_N diverge to $-\infty$. Fix $M<0$ arbitrary. Since $\lim_{n\to\infty} s_n = -\infty$, $\exists N_0$ s.t. $s_n < M-1$ $\forall n > N_0$.

Thus by ①, $a_{N_0} \le M-1$. Since a_N is decreasing and ②, $b_N \le a_N \le M-1 < M$, $\forall N \ge N_0$.

This shows

$\liminf_{n\to\infty} s_n = \lim_{N\to\infty} b_N = \limsup_{n\to\infty} s_n = \lim_{N\to\infty} a_N = -\infty$.

$\boxed{\text{Case 2}}$ $\lim_{n\to\infty} s_n = +\infty$. Then $\lim_{n\to\infty} -s_n = -\infty$.

By Case 1, $\limsup_{n\to\infty} -s_n = \liminf_{n\to\infty} -s_n = -\infty$.

$$\underset{\text{④}\,\|}{\limsup_{n\to\infty} -s_n} \qquad \underset{\text{④}\,\|}{\liminf_{n\to\infty} -s_n}$$

$$-\liminf_{n\to\infty} s_n \qquad\qquad -\limsup_{n\to\infty} s_n$$

Therefore, $\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = +\infty$.

$\boxed{\text{Case 3}}$ $\lim_{n\to\infty} s_n = s$, for $s \in \mathbb{R}$. WTS a_N and b_N converge to s. Fix $\varepsilon > 0$. Then $\exists$ N_0 s.t. $n > N_0$ ensures $|s_n - s| < \frac{\varepsilon}{2}$, or equivalently,

$$\underbrace{s - \frac{\varepsilon}{2} < s_n}_{s-\varepsilon \text{ is a lower bound for } \{s_n : n > N_0\}} \overbrace{< s + \frac{\varepsilon}{2}}^{s+\varepsilon \text{ is an upper bound for } \{s_n : n > N_0\}}$$

$\Downarrow$ $a_N \overset{\text{decreasing}}{\leq} a_{N_0} \leq s + \frac{\varepsilon}{2}$ for any $N \geq N_0$

$\Rightarrow s - \frac{\varepsilon}{2} \leq b_{N_0} \overset{\text{increasing}}{\leq} b_N$ for any $N \geq N_0$

Therefore, $\forall\ N \geq N_0$,

$$s - \varepsilon < s - \frac{\varepsilon}{2} \leq b_N \leq a_N \leq s + \frac{\varepsilon}{2} < s + \varepsilon$$

Hence, $|b_N - s| < \varepsilon$, $|a_N - s| < \varepsilon$.

Therefore

$$\liminf_{n\to\infty} s_n = \lim_{N\to\infty} b_N = \limsup_{n\to\infty} s_n = \lim_{N\to\infty} a_N = s.$$

Now, suppose
$\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$.
WTS $\lim_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$.

$\boxed{\text{Case 1}}$ $\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = -\infty$.
WTS s_n diverges to $-\infty$. Fix $M<0$.
Then $\exists$ N_0 s.t. $N>N_0$ ensures

$$\sup\{s_n : n>N\} = a_N < M.$$

Thus $s_n < M$ for $n>N$ and $N>N_0$.
Hence $s_n<M$ for $n>N_0+1$.
Therefore $\lim_{n\to\infty} s_n = -\infty$.

$\boxed{\text{Case 2}}$ $\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = +\infty$.
Then $\liminf_{n\to\infty} -s_n = \limsup_{n\to\infty} -s_n = -\infty$.
So by Case 1, $\lim_{n\to\infty} -s_n = -\infty$.
Hence (4) ensure $\lim_{n\to\infty} s_n = +\infty$.

$\boxed{\text{Case 3}}$ $\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = s$ for $s \in \mathbb{R}$

WTS s_n converges to s. Fix $\varepsilon > 0$.

Then $\exists\ N_b, N_a$ s.t.

$$\underset{\substack{\|\\ b_N}}{\inf\{s_n : n > N\}}$$

$N > N_b \implies s - \varepsilon < b_N < s + \varepsilon$

$N > N_a \implies s - \varepsilon < a_N < s + \varepsilon$

$$\overset{\substack{a_N\\ \|}}{\sup\{s_n : n > N\}}$$

Thus, for $n > N$ and $N > \max\{N_a, N_b\}$

$s - \varepsilon < b_N \leq s_n \leq a_N < s + \varepsilon$.

Hence, for $n > \max\{N_a, N_b\} + 1$,

$$s - \varepsilon < s_n < s + \varepsilon.$$

Thus, $\lim_{n\to\infty} s_n = s$. $\square$