Lecture 8

CS 117, S25

Announcements:

- No office hours on Fri, May 2

Recall:

Thm: Given a sequence s_n,

MAJOR THM #4

$\lim_{n\to\infty} s_n$ exists $\iff \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$.

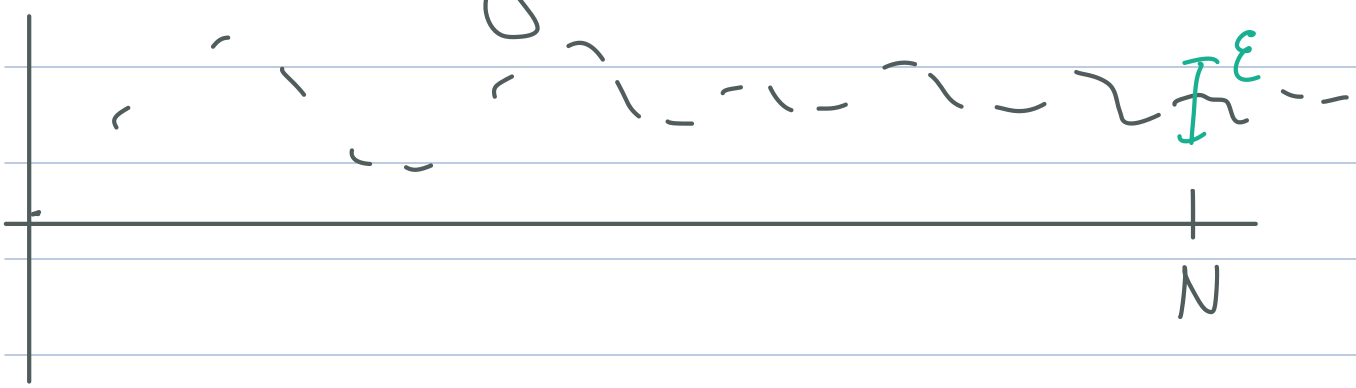
Furthermore, if either of these equivalent conditions holds,

$$\lim_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n.$$

Another important type of sequence...

Def: A sequence s_n is a <u>Cauchy sequence</u> if $\forall\, \varepsilon > 0$, $\exists\, N \in \mathbb{R}$ s.t. $n, m > N$ ensures $|s_n - s_m| < \varepsilon$.

Mental image:



A Cauchy sequence "bunches up" or "gets close and stays close to itself".

How do Cauchy sequences fit in with sequences we already know?

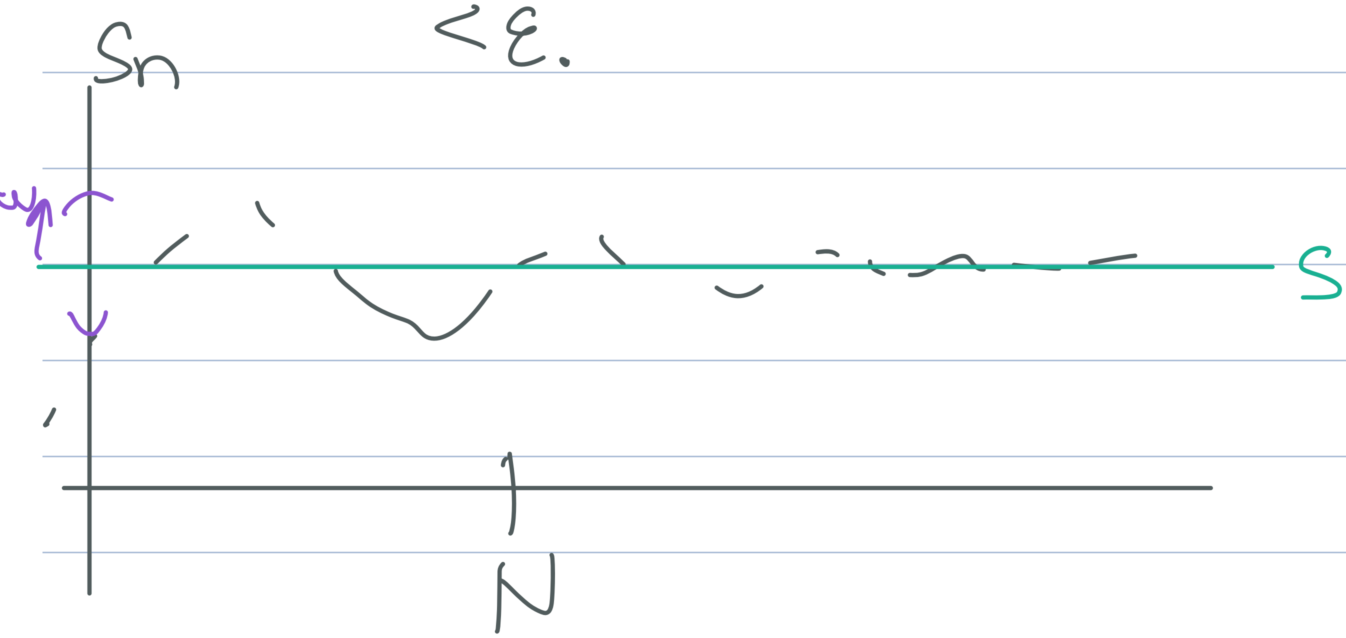
Lemma: Convergent sequences are Cauchy sequence.

Pf: Suppose s_n is a convergent sequence with limit $s \in \mathbb{R}$.
Fix $\varepsilon > 0$ arbitrary.

Since s_n converges to s, $\exists\, N$ s.t. $n > N$ ensures $|s_n - s| < \frac{\varepsilon}{2}$.
Then for $m, n > N$,

$$\begin{aligned} |s_n - s_m| &\le |s_n - s + s - s_m| \\ &\le |s_n - s| + |s - s_m| \\ &< \varepsilon. \end{aligned}$$

$\square$



Lemma: Cauchy sequences are bounded.

Very similar proof to fact that convergent sequences are bdd.

Recall:

s_n is bounded if $\exists\ M > 0$ s.t. $\forall n \in \mathbb{N}$

$|s_n| \leq M \iff -M \leq s_n \leq M$.

Reverse triangle ineq:

$|a| - |b| \leq |a-b|$

Pf: Suppose s_n is a Cauchy sequence. Fix $\varepsilon = \sqrt{5}$. Then $\exists N$ s.t. $n, m > N$ ensures $|s_n - s_m| < \sqrt{5}$. By the reverse triangle inequality, this implies $|s_n| - |s_m| \leq |s_n - s_m| < \sqrt{5}$.

Thus, $n, m > N$ ensures
$|s_n| < |s_m| + \sqrt{5}$.

In particular, taking $m = \lceil N \rceil + 1 \in \mathbb{N}$,
$$|s_n| < \underbrace{|s_{\lceil N \rceil + 1}| + \sqrt{5}}_{:= M_0}, \quad \forall n > N.$$

Define $M := \max\{|s_1|, |s_2|, \dots, |s_{\lceil N \rceil}|, M_0\}$.

Then $|s_n| \leq M \quad \forall n \in \mathbb{N}$. $\square$

don't need a guess for limit; just need to show it bunches up around itself

Thm: A sequence is Cauchy if and only if it is convergent.

here, you must have a guess of the limit and show the sequence gets close to the limit

Recall:

If $s_n \le t_n$ for all but finitely many n and limits exist, then $\lim_{n\to\infty} s_n \le \lim_{n\to\infty} t_n$.

HW4: $r_n \le t_n \ \forall n$ and limits exist $\Rightarrow \lim_{n\to\infty} r_n \le \lim_{n\to\infty} t_n$

If $a \le b + \varepsilon$ for $\forall \varepsilon > 0$, then $a \le b$.

- Claim:

$$||a|-|b|| \le |a-b| \quad \forall a,b \in \mathbb{R}$$

$$\Updownarrow$$

$$|a|-|b| \le |a-b| \quad \forall a,b \in \mathbb{R}$$

Pf of Claim: Assume top is true. Since $x \le |x| \ \forall x \in \mathbb{R}$,

$$\underbrace{|a|-|b|}_{x} \le \underbrace{||a|-|b||}_{|x|} \le |a-b|.$$

Assume bottom is true.

Fix $x, y \in \mathbb{R}$. WTS $||x|-|y|| \le |x-y|$.

First, taking $a=x, b=y$, we have

$$\underbrace{|x|-|y|}_{A} \le \underbrace{|x-y|}_{B}.$$

Next, taking $a=y, b=x$, we have

$$\underbrace{|y|-|x|}_{-A} \le |y-x| = \underbrace{|x-y|}_{B}$$

Finally, if $A \le B$ and $-A \le B$, we must have $|A| \le B$.

Therefore $||x|-|y|| \le |x-y|$.

Pf: We have already shown that convergent sequences are Cauchy. Now suppose s_n is Cauchy, and we will show it is convergent.

To show s_n converges, it suffices

to show that

$$\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n.$$

$$\lim_{N\to\infty} \underbrace{\inf\{s_n : n>N\}}_{b_N} \qquad \lim_{N\to\infty} \underbrace{\sup\{s_n : n>N\}}_{a_N}$$

This ensures $\lim_{n\to\infty} s_n$ exists.

Since all Cauchy sequences are bounded, s_n can't diverge to $\pm\infty$, so we would have $\lim_{n\to\infty} s_n \in \mathbb{R}$, so s_n converges.

Fix $\varepsilon>0$. Since s_n is Cauchy, $\exists N$ s.t. $n,m>N$ ensures $|s_n - s_m| < \varepsilon$

$\iff s_m - \varepsilon < s_n < s_m + \varepsilon.$

Thus, for $m>N$, we have

$a_N = \sup\{s_n : n>N\} \le s_m + \varepsilon.$

$\iff a_N - \varepsilon \le s_m.$

Likewise, we have

$$a_N - \varepsilon \le \inf\{s_m : m > N\} = b_N$$

Since a_N is a decreasing sequence and b_N is an increasing sequence, $\forall\, k > N$,

$$\underbrace{a_K - \varepsilon}_{\substack{\text{limit exists} \\ \text{b/c decreasing} \\ \text{sequence}}} \le a_N - \varepsilon \le b_N \le \underbrace{b_K}_{\substack{\text{limit exists} \\ \text{b/c increasing}}}.$$

Thus, by ,

$$\limsup_{n\to\infty} s_n - \varepsilon = \lim_{k\to\infty} a_K - \varepsilon \le \lim_{k\to\infty} b_K = \liminf_{n\to\infty} s_n.$$

Since this holds for $\varepsilon > 0$ arbitrary, by , $\limsup_{n\to\infty} s_n \le \liminf_{n\to\infty} s_n$.
Since we always have $\limsup_{n\to\infty} s_n \ge \liminf_{n\to\infty} s_n$, equality must hold.