Lecture 9
CS 117, S25
© Katy Craig, 2025

Announcements:

- Makeup lecture on Friday, May 9th, 11am - 12:15pm

Recall:

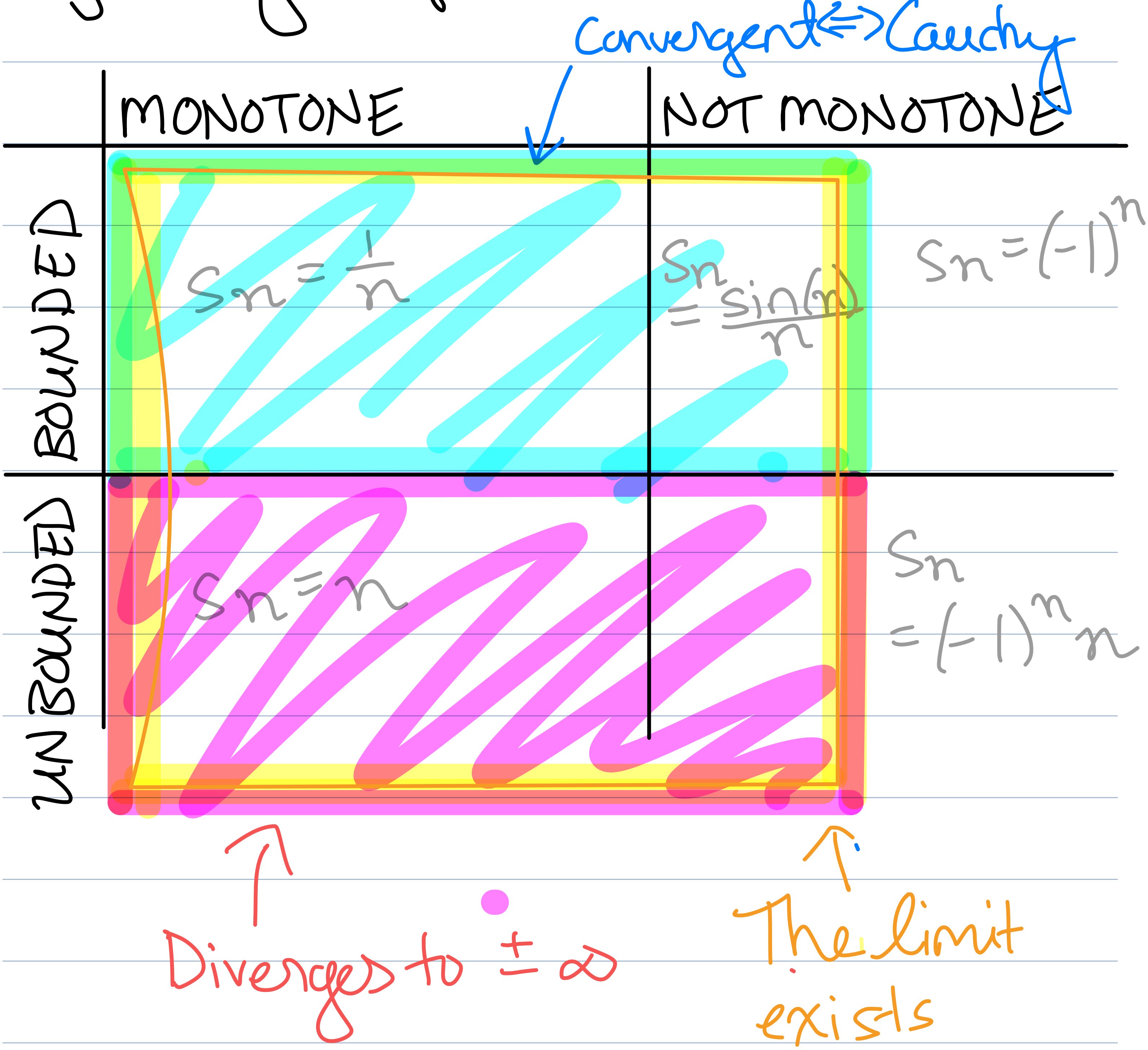
Def: A sequence is a **Cauchy sequence** if $\forall\, \varepsilon > 0$, $\exists\, N \in \mathbb{R}$ s.t. $n, m > N$ ensures $|s_n - s_m| < \varepsilon$.

Lemma: Convergent sequences are Cauchy sequences

Lemma: Cauchy sequences are bounded

Thm: A sequence is Cauchy if and only if it is convergent.

Types of Sequences



	MONOTONE	NOT MONOTONE
BOUNDED	$S_n = \frac{1}{n}$	$S_n = \frac{\sin(n)}{n}$ $S_n = (-1)^n$
UNBOUNDED	$S_n = n$	$S_n = (-1)^n n$

Convergent ⇔ Cauchy

Diverges to ± ∞

The limit exists

We already know a lot about monotone sequences...

Now, we will prove an important theorem about bounded sequences... via the notion of a subsequence.

Recall:

Def: A sequence is a function whose domain is a set of the form $\{m, m+1, m+2, \ldots\}$ for some $m \in \mathbb{Z}$. We study sequences whose range is $\mathbb{R}$.

We write s_n instead of $s(n)$, to emphacise it's a special type of fn.

Def: Given a sequence $s_n, n \in \mathbb{N}$, and a strictly increasing sequence n_k of natural numbers, a sequence of the form s_{n_k} is a **subsequence** of s_n.

Remark: Just like $s(n)$ is equivalent to s_n, $s(n(k))$ is equivalent to s_{n_k}.

$$\text{Ex: } s_n = (\overset{n=1}{-1}, \overset{n=2}{2}, \overset{n=3}{-3}, 4, -5, \dots)$$

$$n_k = (\overset{k=1}{1}, \overset{k=2}{3}, \overset{k=3}{5}, 7, 9, \dots)$$

$$s_{n_k} = (\overset{k=1}{-1}, \overset{k=2}{-3}, \overset{k=3}{-5}, \dots)$$

$a_N = \sup\{s_n : n > N\} = (\overset{N=1}{+\infty}, \overset{N=2}{+\infty}, \overset{N=3}{+\infty}, \ldots$

$b_N = \inf\{s_n : n > N\} = (-\infty, -\infty, -\infty, \ldots$

the sequence of indices n_K is infinite

Informally, a subsequence s_{n_K} of s_n is any infinite collection of elements from the original sequence, listed in order.

the sequence of indices n_K is strictly increasing, $n_K < n_{K+1}$

Limits of Subsequences

Lemma: For any strictly increasing sequence of natural numbers n_K, we have $n_K \geq K \ \forall K \in \mathbb{N}$.

Pf: by induction :)

Def: A subsequential limit of a sequence s_n is a real number or symbol $\pm\infty$ that is the limit of some subsequence of s_n.

Ex: $s_n = (1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, \ldots)$
0 and $+\infty$ are subsequential limits. In fact, these are the only subsequential limits.

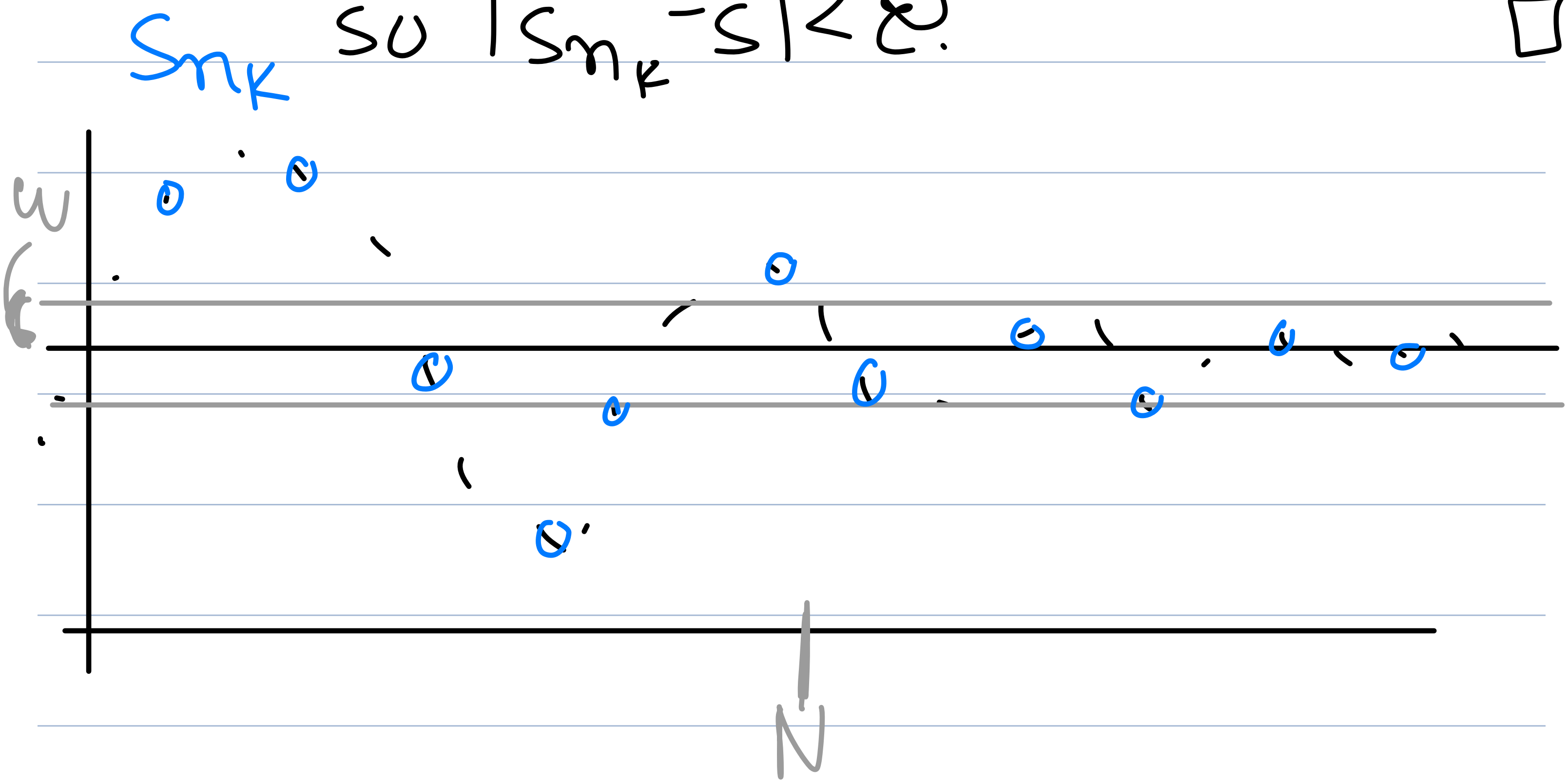
Thm: If a sequence s_n converges to $s \in \mathbb{R}$, then every subsequence also converges to s.

Rmk: A consequence of this theorem is if a sequence s_n converges to s, then s is the only subsequential limit.

Pf: Fix an arbitrary subsequence s_{n_k}. We must show $\lim_{k\to\infty} s_{n_k} = s$.
Fix $\varepsilon > 0$. Since s_n converges to s, $\exists\, N \in \mathbb{R}$ s.t. $n \geq N$ ensures

$$|s_n - s| < \varepsilon.$$

Thus, if $k \geq N$, by Lemma, $n_k \geq k \geq N$ so $|s_{n_k} - s| < \varepsilon$. $\square$



Thm (main subsequence theorem)

Let s_n be a sequence.

(a) For any $t \in \mathbb{R}$

[t is a subsequential limit of s_n]

$\Updownarrow$

[the set $\{n : |s_n - t| < \varepsilon\}$ is infinite, for all $\varepsilon > 0$]

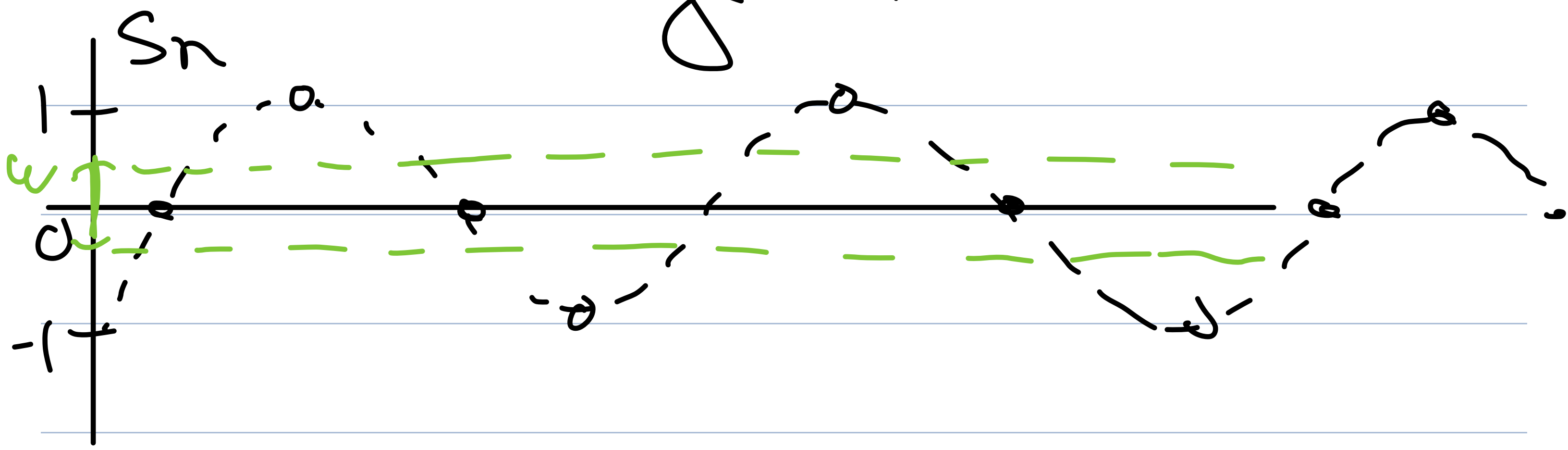
$\Updownarrow$ ← another way of writing same thing

[$\forall \varepsilon > 0$, $|\{n : |s_n - t| < \varepsilon\}| = +\infty$]

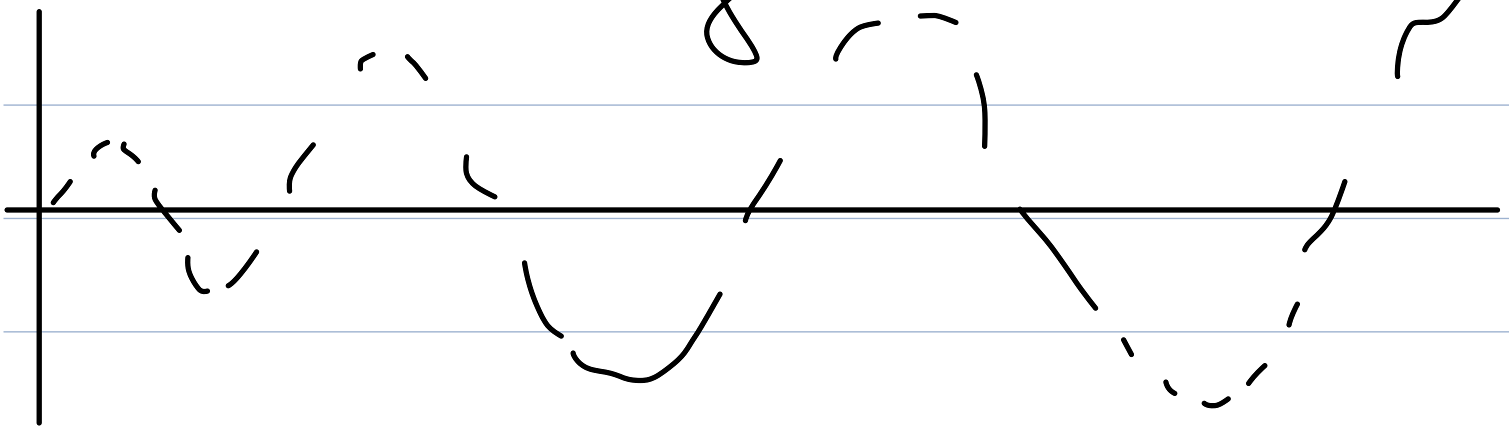
(b) $+\infty$ is a subsequential limit $\iff$ s_n is unbounded above

(c) $-\infty$ is a subsequential limit $\iff$ s_n is unbounded below

Mental image (a)



Mental image (b) + (c)



Lemma: If s_n is unbounded above, then $\forall M>0$, $|\{n : s_n > M\}| = +\infty$.

Pf: Assume for the sake of contradiction, that $\exists M > 0$ s.t. $|\{n : s_n > M\}| < +\infty$.

Then $|\{s_n : s_n > M\}| \underset{n \mapsto s_n \text{ is surjective}}{\le} |\{n : s_n > M\}|$

Let $s_{max} = \max\{s_n : s_n > M\}$

Then $s_n \le s_{max}$ if $n \in \{n : s_n > M\}$
$s_n \le M \le s_{max}$ if $n \in \{n : s_n \le M\}$

This shows s_n is bounded above by s_{max}, which is a contradiction.