Midterm 2 Solutions

①

ⓐ By the main subsequences theorem, it suffices to show that, $\forall \varepsilon>0$, $|\{n : |s_n - s| < \varepsilon\}| = +\infty$.

Assume, for the sake of contradiction that $\exists\ \varepsilon>0$ s.t. $|\{n : |s_n - s| < \varepsilon\}| < +\infty$. Then $N_\varepsilon := \max\{n : |s_n - s| < \varepsilon\}$ exists. Since $|s_n - s| < \varepsilon \iff s-\varepsilon < s_n < s+\varepsilon$ and we have $s_n < s+\varepsilon\ \forall n \in \mathbb{N}$, for $n > N_\varepsilon$, $s_n \leq \underbrace{s-\varepsilon}_{M_1}$.

On the other hand, for $n \leq N_\varepsilon$, $s_n \leq \underbrace{\max\{s_k : k \leq N_\varepsilon\}}_{M_2} < s$.

Thus, $M := \max\{M_1, M_2\}$ is an upper bound for $\{s_n : n \in \mathbb{N}\}$ that is strictly less than s. This is a contradiction, since s is the least upper bound.

(b) Consider $s_n = \frac{1}{n}$. Then $s = 1 = s_1$, but since $\lim_{n\to\infty} s_n = 0$, all subsequences of s_n converge to zero

(2) (a)

First, note that, for any $n > N$,
$s_n + t_n \leq \sup\{s_n : n > N\} + \sup\{t_n : n > N\}$.
Hence, the RHS is an upper bound for $\{s_n + t_n : n > N\}$, so

$$\overbrace{\sup\{s_n + t_n : n > N\}}^{a_N}$$
$$\leq \underbrace{\sup\{s_n : n > N\}}_{b_N} + \underbrace{\sup\{t_n : n > N\}}_{c_N}.$$

Since b_N and c_N are convergent, the limit of the RHS is
$\lim b_N + \lim c_N = \limsup s_n + \limsup t_N$.
Since a_N is convergent, the limit of the LHS is $\limsup s_n + t_n$. This gives the result.

(b) $s_n = (-1)^n$, $t_n = (-1)^{n+1}$.

$\limsup s_n + t_n = 0 < 2 = \limsup s_n + \limsup t_n$

(3)

(a) Since $s_n \geq 0 \ \forall n \in \mathbb{N}$, $s \geq 0$.

First, suppose $s > 0$. Then there exists N_0 s.t. $n > N_0$ ensures $s_n > \frac{s}{2} \Leftrightarrow \sqrt{s_n} > \sqrt{\frac{s}{2}}$

Fix $\varepsilon > 0$. Note that, for $n > N_0$,

$$|\sqrt{s_n} - \sqrt{s}| < \varepsilon \Leftrightarrow \frac{|\sqrt{s_n} - \sqrt{s}|(\sqrt{s_n} + \sqrt{s})}{\sqrt{s_n} + \sqrt{s}} < \varepsilon$$

$$\Leftrightarrow |s_n - s| < (\sqrt{s_n} + \sqrt{s})\varepsilon$$

$$\Leftarrow |s_n - s| < \left(\sqrt{\tfrac{s}{2}} + \sqrt{s}\right)\varepsilon$$

Since $s_n \to s$, $\exists$ N_1 s.t. $n > N_1$ ensures $|s_n - s| < \left(\sqrt{\frac{s}{2}} + \sqrt{s}\right)\varepsilon$. Thus $n > \max\{N_0, N_1\}$ ensures $|\sqrt{s_n} - \sqrt{s}| < \varepsilon$.

Next, suppose $s=0$. Fix $\varepsilon>0$.
Note that $|\sqrt{s_n}|<\varepsilon \Leftrightarrow s_n<\varepsilon^2$.
$\Leftrightarrow |s_n-s|<\varepsilon^2$. Since $s_n \to s$, $\exists N$ s.t.
$n>N$ ensures $|s_n-s|<\varepsilon^2 \Rightarrow |\sqrt{s_n}-\sqrt{s}|<\varepsilon$.

This shows $\sqrt{s_n} \to \sqrt{s}$.

(b) Fix $x_0 \in [0,+\infty)$. Consider a sequence x_n of nonnegative numbers x_n s.t. $x_n \to x_0$. By (a), $f(x_n) \to f(x_0)$. Thus f is continuous at x_0. Since $x_0 \in [0,+\infty)$ was arbitrary, f is continuous on $[0,+\infty)$.

(c) Define $\varphi(x) = \frac{x^{100}}{\sqrt{1-x}}$. Note that $\varphi(x) = g(x)/f(h(x))$. Since h is cts on $[0,1]$ and f is cts on $h([0,1])=[0,1]$, $f \circ h$ is cts on $[0,1]$. Furthermore, since $f \circ h$ is nonzero on $[0,1)$, and g is cts on $[0,1)$, φ is cts on $[0,1)$.

Note that $\varphi(0)=0$. Since $g(x)$ is continuous and $g(1)=1$, $\exists\ \delta_0>0$ s.t. $x\in(1-\delta_0,1)$ ensures $g(x)>\frac{1}{2}$. Since $f(h(x))$ is continuous on $[0,1]$ and $f(h(1))=0$, $\exists\ \delta_1>0$ s.t. $x\in(1-\delta_1,1)$ ensures $f(h(x))<\frac{1}{4}$. Taking $\delta=\min(\delta_0,\delta_1)$, $x_0\in(1-\delta,1)$ ensures $\varphi(x_0)=\frac{g(x_0)}{f(h(x_0))}>\frac{4}{2}=2$.

Thus, by the intermediate value theorem, for all $c\in[0,2]$ $\exists\ y\in[0,x_0]$ s.t. $\varphi(y)=c$. In the case $c=2$, we see $y=0$ is impossible ($\varphi(0)=0$), so $y\in(0,x_0]\subseteq(0,1)$.

4 a First, note that, $\forall n \in \mathbb{N}$.
$|a_{n+1} - a_n| \leq \alpha^{n-1} |a_2 - a_1|$

Thus, by the triangle inequality, $\forall m \geq n$,

$$|a_m - a_n| = |a_m - a_{m-1} + a_{m-1} - \dots + a_{n+1} - a_n|$$
$$\leq \sum_{k=n}^{m-1} |a_{k+1} - a_k|$$
$$\leq \sum_{k=n}^{m-1} \alpha^{k-1} |a_2 - a_1|$$

Since $\alpha \in [0,1)$, the series $\sum_{k=1}^{\infty} \alpha^k$ is convergent. Thus, by the Cauchy criterion, $\forall \varepsilon > 0$, $\exists\, M$ s.t. $m \geq n \geq M$ ensures

$$\sum_{k=n}^{m-1} \alpha^{k-1} |a_2 - a_1| < \varepsilon.$$

This gives the result

(b) Since a_n is Cauchy, it converges to some $a \in \mathbb{R}$. We also have a_{n+1} converges to a.
Since f is cts, $a_{n+1} = f(a_n)$ ensures

$$a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} f(a_n) = f(a).$$

Suppose there exists $b \neq a$ so that $f(b) = b$. Then

$$\underbrace{|f(a) - f(b)|}_{= |a-b|} \leq \alpha |a-b| < |a-b|$$

This is a contradiction. Thus, the fixed point is unique.