Midterm 1 Solutions

① Since $L<1$, if we define $a=\frac{L+1}{2}$, then $0\leq L<a<1$. Let $\varepsilon=a-L$. Then $\exists$ N s.t. $\forall n>N$, $\left|\left|\frac{s_{n+1}}{s_n}\right|-L\right|<\varepsilon$

$\implies \left|\frac{s_{n+1}}{s_n}\right|<L+\varepsilon=a \iff \underbrace{|s_{n+1}|<a|s_n|}_{*}$.

We now prove that $|s_n|\leq a^{n-N-1}|s_{N+1}|$ for all $n>N$ using induction. For the base case of $n=N+1$, note that $|s_{N+1}|=a^0|s_{N+1}|=a^{n-N-1}|s_{N+1}|$. For the inductive step, assume $|s_n|\leq a^{n-N-1}|s_{N+1}|$. By (*) above, this implies $|s_{n+1}|\leq a|s_n|\leq a^{n+1-N-1}$, which completes the proof of the inductive step.

Finally, we use that $|s_n|\leq a^{n-N-1}|s_{N+1}|$ to prove that $\lim_{n\to\infty}s_n=0$. Since $a<1$, $\lim_{n\to\infty}a^n=0$. Since the limit of the product is the product of the limit, $\lim_{n\to\infty}a^n a^{-N-1}|s_{N+1}|=0$ $\forall N\in\mathbb{N}, n>N$.

Fix $\varepsilon>0$ and $N\in\mathbb{N}$. Choose $\tilde{N}$ s.t. $n>\tilde{N}$ ensures $|a^{n-N-1}|s_{N+1}||<\varepsilon$. Then $n>\tilde{N}$ ensures $|s_n|<\varepsilon$. This shows $\lim_{n\to\infty} s_n=0$.

(2) If $\lim_{n\to\infty} a_n>0$, either a_n diverges to $+\infty$ or converges to some $a>0$. In the former case, $\exists$ N s.t. $n>N$ ensures $a_n>1$. In the latter case, $\exists$ $\tilde{N}$ s.t. $n>N$ ensures $|a_n-a|<\frac{a}{2}$ $\Rightarrow$ $a_n>a-\frac{a}{2}=\frac{a}{2}>0$. In both cases, we see $\exists$ $b>0$ s.t. $a_n\geq b$ for all but finitely many n.

Applying this to $a_n=s_nt_n$ and $a_n=t_n$, $\exists$ $b_1, b_2>0$ so that $t_n\geq b_1$, $s_nt_n\geq b_2$ for all but finitely many n.

Thus, there exists N_1, N_2 so that $n>N_1$ ensures $t_n\geq b_1\geq 0$ and $n>N_2$ ensures $t_ns_n\geq b_2\geq 0$. Thus $n>\max\{N_1,N_2\}$ ensures $s_n\geq 0$. This shows $\{n\in\mathbb{N}: s_n\leq 0\}$ has at most $\max\{N_1,N_2\}$ elements.

③ See HW2, Q9

④ This was a problem on our Ph.D. students' qualifying exam $\ddot\smile$.

ⓐ Suppose s_n converges to s. Then s_n is bounded so $\exists\, M$ s.t. $|s_n - s| \leq M \;\forall n$

Fix $\varepsilon > 0$. $\exists\, N$ s.t. $m > N$ ensures $|s_m - s| < \frac{\varepsilon}{2}$

Thus, if $n > (N+1)^2$, $\lfloor \sqrt{n} \rfloor > N$ and

$$
\begin{aligned}
|\sigma_n - s| &= \left| \frac{s_1 + s_2 + \dots + s_n}{n} - s \right| = \left| \frac{(s_1 - s)}{n} + \dots + \frac{(s_n - s)}{n} \right| \\
&\leq \left| \frac{s_1 - s}{n} \right| + \dots + \left| \frac{s_n - s}{n} \right| \\
&\leq \frac{M}{n} \cdot \lfloor \sqrt{n} \rfloor + \left| \frac{s_{\lfloor \sqrt{n} \rfloor} - s}{n} \right| + \left| \frac{s_{\lfloor \sqrt{n} \rfloor + 1} - s}{n} \right| + \dots + \left| \frac{s_n - s}{n} \right| \\
&\leq \frac{M}{\sqrt{n}} + \frac{\varepsilon \cdot n}{2n} \\
&= \frac{M}{\sqrt{n}} + \frac{\varepsilon}{2}
\end{aligned}
$$

Finally, choosing $\tilde{N} = \max\left\{ \left(\frac{M}{\varepsilon}\right)^2, (N+1)^2 \right\}$, we have that $n > \tilde{N}$ ensures $|\sigma_n - s| < \varepsilon$. This shows $\sigma_n \to s$.

ⓑ Let $s_n = (-1)^n$. Then $\sigma_n = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{-1}{n} & \text{if } n \text{ odd.} \end{cases}$

Hence σ_n converges but s_n doesn't.

(c) Following the hint, we see

$$\frac{1}{n+1}\sum_{k=1}^{n} k a_k = \frac{1}{n+1}\sum_{k=1}^{n} k(s_{k+1}-s_k)$$

$$= \frac{1}{n+1}\left[(s_2-s_1)+2(s_3-s_2)+\dots+n(s_{n+1}-s_n)\right]$$

$$= \frac{1}{n+1}\left[-s_1-s_2-s_3-s_n+\overbrace{n s_{n+1}}^{-s_{n+1}+(n+1)s_{n+1}}\right]$$

$$= s_{n+1} - \frac{s_1+s_2+s_3+\dots+s_n+s_{n+1}}{n+1}$$

which gives the result.

Since $k a_k \to 0$, part (a) ensures $\frac{1}{n}\sum_{k=1}^{n} k a_k \to 0$, so $\left(\frac{n}{n+1}\right)\cdot\frac{1}{n}\sum_{k=1}^{n} k a_k \to 0$.

Since σ_{n+1} converges, this implies s_{n+1} converges to the same limit.

Q3) ⓐ Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = s$, for all $\varepsilon > 0$, there exists N_a and N_b so that $n > N_a$ ensures $|a_n - s| < \varepsilon$ and $n > N_b$ ensures $|b_n - s| < \varepsilon$. Recall that $|a_n - s| < \varepsilon \Leftrightarrow s - \varepsilon < a_n < s + \varepsilon$ and $|b_n - s| < \varepsilon \Leftrightarrow s - \varepsilon < b_n < s + \varepsilon$. Define $N = \max\{N_a, N_b\}$. Then for all $\varepsilon > 0$, $n > N$ ensures $s - \varepsilon < a_n$ and $b_n < s + \varepsilon$. Since $a_n \leq s_n \leq b_n$, this shows that for all $\varepsilon > 0$, $n > N$ ensures $s - \varepsilon < s_n < s + \varepsilon$ $\Leftrightarrow |s_n - s| < \varepsilon$, i.e. $\lim_{n\to\infty} s_n = s$.

ⓑ Apply the squeeze lemma, since

$$-t_n \leq s_n \leq t_n$$

and $\lim_{n\to\infty} t_n = 0$, $\lim_{n\to\infty} -t_n = (-1)(0) = 0$.

ⓒ If $\lim_{n\to\infty} s_n = 0$, it is not necessarily true that $\lim_{n\to\infty} t_n = 0$. For example, if $s_n = 0$ $\forall n \in \mathbb{N}$ and $t_n = 1$ $\forall n \in \mathbb{N}$, then $|s_n| \leq t_n$ $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n = 0$, but $\lim_{n\to\infty} t_n = 1 \neq 0$.

4

(a) Fix $\varepsilon > 0$. Note that, by the reverse triangle inequality,

$$||t| - |t_n|| \leq |t - t_n|.$$

Since $\lim_{n\to\infty} t_n = t$, $\exists\ N \in \mathbb{R}$ s.t. $n > N$ ensures $|t - t_n| < \varepsilon$, hence $||t| - |t_n|| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this shows $\lim |t_n| = |t|$.

(b) The converse is not true.
Let $t_n = (-1)^n$. Then $|t_n| = 1$ is a convergent sequence, but t_n is not a convergent sequence.

5 (a) First, suppose $\lim_{n\to\infty} s_n = s$ for $s < a$. If we define $\varepsilon = a - s$, then $\varepsilon > 0$. By definition of convergence, there exists $N \in \mathbb{R}$ s.t. $n > N$ ensures $|s_n - s| < \varepsilon \iff s - \varepsilon < s_n < s + \varepsilon$

Thus $n>N$ ensures $s_n<s+\varepsilon=a$.
Hence $\{n\in\mathbb{N} : s_n \geq a\} \subseteq \{1,2,\dots,\lfloor N\rfloor\}$.
Thus, $\{n\in\mathbb{N}: s_n\geq a\}$ is a finite set.

Next, suppose $\lim_{n\to\infty} s_n = -\infty$. Let $m=\min\{-1,a\}$. Then $\exists$ N s.t. $s_n<m\leq a$ for all $n>N$.
Thus, $\{n\in\mathbb{N}: s_n\geq a\}$ is a finite set.

(b) First, suppose $\lim_{n\to\infty} t_n = t>0$. Then $\lim_{n\to\infty} t_n > \frac{t}{2}$. Applying part (a) with $s_n=-t_n$ and $a=-\frac{t}{2}$, we obtain that
$\{n\in\mathbb{N}: s_n\geq a\} = \{n\in\mathbb{N}: -t_n\geq -\frac{t}{2}\}$
$=\{n\in\mathbb{N}: t_n\leq \frac{t}{2}\} \supseteq \{n\in\mathbb{N}: t_n<\frac{t}{2}\}$
are all finite sets. This shows the result for $b=\frac{t}{2}$.

Now, suppose $\lim_{n\to\infty} t_n = +\infty$. Then $\exists$ N s.t. $\forall n>N$, $t_n>1$. Thus, $\{n\in\mathbb{N}: t_n\leq 1\}\supseteq\{n\in\mathbb{N}: t_n<1\}$ is finite. This shows the result for $b=1$.

(6) Suppose $\lim_{n \to \infty} s_n = s$

(a) Fix $\varepsilon > 0$. Since s_n converges to s, there exists N s.t. $n > N$ ensures

$$|s_n - s| < \varepsilon.$$

Let $\tilde{N} = \max\{N, m\}$. Then $n > \tilde{N}$ ensures

$$|t_n - s| = |s_n - s| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this shows $\lim_{n\to\infty} t_n = s = \lim_{n\to\infty} s_n$.

(b) Fix $\varepsilon > 0$. Since s_n converges to s, there exists N s.t. $n > N$ ensures

$$|s_n - s| < \varepsilon.$$

Since $n > N$ implies $n+m > N$, we have

$$|t_n - s| = |s_{n+m} - s| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this shows $\lim_{n\to\infty} t_n = s = \lim_{n\to\infty} s_n$.

(8) By the inequality,

$$|t_n^p - t^p| \le p \max\{(t_n)^{p-1}, (t)^{p-1}\} |t_n - t|$$

Since t_n is a convergent sequence, it is bounded and $\exists$ $M>0$ s.t. $|t_n|<M$ for all n. By Question 5, $\lim_{n\to\infty} |t_n| = |t|$. Let $\tilde{M} = \max(M, |t|)$. Then the above inequality ensures

$$(*) \; |t_n^p - t^p| \le p \tilde{M}^{p-1} |t_n - t|.$$

Fix $\varepsilon>0$. Since t_n converges to t, $\exists$ N s.t. $n>N$ ensures $|t_n - t| < \frac{\varepsilon}{p \tilde{M}^{p-1}}$.

Then, by $(*)$, $n>N$ ensures $|t_n^p - t^p| < \varepsilon$. Since $\varepsilon>0$ was arbitrary, this shows $\lim t_n^p = t^p$.

(9) The correct definition is (d).

(a) Consider $s_n = (1, -1, 1, -1, \dots)$, $s=0$. Then for $\varepsilon = 2$ and all $N \in \mathbb{R}$, $n>N$ ensures $|s_n - s| < \varepsilon$.

(b) Consider $s_n = \frac{1}{n}, s = 0$. For $\varepsilon = 0$ there is no $N \in \mathbb{R}$ so that $n > N$ ensures $|s_n - 0| < \varepsilon$.

(c) Consider $s_n = \frac{1}{n}, s = 0$. For $\varepsilon = \frac{1}{4}$, $|s_n - s| < \varepsilon$ is not true for all $n \in \mathbb{N}$.

(10) If $\lim_{n\to\infty} r_n = -\infty$, the result is immediate. Thus, it remains to consider the remaining cases.

Case 1: Suppose $\lim_{n\to\infty} r_n = r \in \mathbb{R}$

Case 1a: If $\lim_{n\to\infty} t_n = +\infty$, we are done.

Case 1b: Suppose $\lim_{n\to\infty} t_n = t \in \mathbb{R}$. Assume for the sake of contradiction that $t < r$. Let $\varepsilon = \frac{r-t}{2} > 0$. Then $\exists N_r, N_t$ s.t. $n > N_r$ ensures $|r_n - r| < \varepsilon$ and $n > N_t$ ensures $|t_n - t| < \varepsilon$. Let $N = \max\{N_r, N_t\}$. Then $n > N$ ensures

$$t_n < t + \varepsilon = t + \frac{r-t}{2} = \frac{t+r}{2} = r - \frac{r-t}{2} = r - \varepsilon < r_n.$$

This contradicts that $r_n \leq t_n \ \forall n \in \mathbb{N}$. Thus $\lim_{n\to\infty} t_n = t \geq r = \lim_{n\to\infty} r_n$.

Case 1c: Suppose $\lim_{n\to\infty} t_n = -\infty$. Then $\exists N_t$ s.t. $\forall n > N_t$, $t_n < r - 1$. There also exists N_r s.t. $\forall n > N_t$, $r - 1 < r_n$. Thus for $N = \max\{N_t, N_r\}$, $n > N$ ensures

$$t_n < r-1 < r_n.$$

Again, this contradicts that $r_n \le t_n \ \forall n \in \mathbb{N}$.
Thus $\lim_{n\to\infty} t_n = -\infty$ is impossible.

Case 2: Suppose $\lim_{n\to\infty} r_n = +\infty$. Fix $M>0$. Then $\exists\, N$ s.t. $\forall\, n \ge N$, $M < r_n \le t_n$. This shows $\lim_{n\to\infty} t_n = +\infty$.

11 Suppose s_n is increasing. Fix $n \in \mathbb{N}$.
We will prove
$$m \ge n \Rightarrow s_m \ge s_n$$
by induction.

Base case: $m=n$. By definition $s_m = s_n$.

Inductive step: Suppose $m \ge n$ and $s_m \ge s_n$. Since it is an increasing sequence, $s_{m+1} \ge s_m \ge s_n$. This shows the inductive step.

Now, suppose $m \ge n \Rightarrow s_m \ge s_n \ \forall\, n, m \in \mathbb{N}$. Take $m = n+1$. Then $s_{n+1} \ge s_n \ \forall n \in \mathbb{N}$. This shows s_n is increasing.

12 a Assume for the sake of contradiction that s_n converges to some $s \in \mathbb{R}$. Then $\exists N$ s.t. $\forall n > N$, $s-1 < s_n < s+1$. Thus, it is impossible for s_n to diverge to $+\infty$

Since, for $M = |s+1|$, there is no N_M s.t. $s_n > M = |s+1| \geq s+1$ $\forall n > N_M$. Likewise, it is impossible for s_n to diverge to $-\infty$ since for $m = -|s-1|$, there is no $\tilde{N}_m$ s.t. $s_n < m = -|s-1| \leq s-1$ $\forall n > \tilde{N}_m$.

b Suppose $\lim_{n\to\infty} s_n = +\infty$. Fix $m < 0$. Then $-m > 0$, so $\exists$ N s.t. $n > N$ ensures $s_n > -m \Rightarrow m > -s_n$. Thus $\lim_{n\to\infty} -s_n = -\infty$.

Now, suppose $\lim_{n\to\infty} -s_n = -\infty$. Fix $M > 0$. Then $-M < 0$, so $\exists$ N s.t. $n > N$ ensures $-s_n < -M \Rightarrow s_n > M$. Thus $\lim_{n\to\infty} s_n = +\infty$.

Case 1: $\lim_{n\to+\infty} s_n \in \mathbb{R}$
Since $t_n = (k, k, k, \dots)$ is a sequence that converges to k and s_n is a convergent sequence, by the theorem that the limit of the product is the product of the limits,

$$\lim_{n\to+\infty} ks_n = \lim_{n\to+\infty} t_n s_n = \left(\lim_{n\to+\infty} t_n\right)\left(\lim_{n\to+\infty} s_n\right) = k \lim_{n\to+\infty} s_n.$$

Case 2:$\lim_{n\to+\infty} s_n = \pm\infty$ and $k = 0$
Then $ks_n = (0, 0, 0, \dots)$ converges to $0 = k \cdot (+\infty) = k \lim_{n\to+\infty} s_n$.

Case 3a: $\lim_{n\to+\infty=+\infty}$ and $k > 0$
We must show that ks_n diverges to $+\infty$. Fix $M > 0$. Since s_n diverges to ∞, there exists N so that $n >$ ensures $s_n > M/k \implies ks_n > M$. This shows $\lim_{n\to+\infty} ks_n = +\infty$.

Case 3b: $\lim_{n\to+\infty=+\infty}$ and $k < 0$
Then $-(ks_n) = (-k)s_n$. By Case 3a, $\lim_{n\to+\infty}(-k)s_n = +\infty$. By Q12(b), this implies $\lim_{n\to+\infty} ks_n = -\infty$.

Case 4a: $\lim_{n\to+\infty} s_n = -\infty$ and $k > 0$
Then $-(ks_n) = k(-s_n)$. By Q12(b), $\lim_{n\to+\infty} -s_n = +\infty$. Thus, Case 3a ensures $\lim_{n\to+\infty} k(-s_n) = +\infty$. Thus, by Q12 again, $\lim_{n\to+\infty} ks_n = -\infty$.

Case 4b: $\lim_{n\to+\infty} s_n = -\infty$ and $k < 0$
Then $-(ks_n) = (-k)s_n$. By Case 4a, $\lim_{n\to+\infty}(-k)s_n = -\infty$. By Q12(b), this implies $\lim_{n\to+\infty} ks_n = +\infty$.

13)