Math CCS 117: Midterm 2

Thursday May 29, 2025

Name: ______________________________

Signature: ______________________________

This is a closed-book and closed-note examination. Please show your work in the space provided. You may use scratch paper. You have 1 hour and 15 minutes.

Question	Points	Score
1	8	
2	7	
3	15	
4	extra credit	
Total	30	

Question 1 (8 points)

Suppose s_n is a bounded sequence, and define $s = \sup\{s_n : n \in \mathbb{N}\}$.

(a) Suppose $s_n < s$ for all $n \in \mathbb{N}$. Prove that there exists a subsequence of s_n converging to s.

(b) Suppose $s_n = s$ for some $n \in \mathbb{N}$. Give an example of such a sequence that doesn't have a subsequence converging to s.

Question 2 (7 points)

Suppose s_n and t_n are bounded sequences.

(a) Prove that $\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n$.

(b) Give an examples of bounded sequences s_n and t_n for which

$$\limsup(s_n + t_n) < \limsup s_n + \limsup t_n.$$

Question 3 (15 points)

(a) Suppose s_n is a nonnegative sequence that converges to s. Prove that $\sqrt{s_n}$ converges to $\sqrt{s}$. (You may use the fact that, for any $a, b \geq 0$, $a \leq b \iff \sqrt{a} \leq \sqrt{b}$.)

(b) Prove that $f(x) = \sqrt{x}$ is a continuous function on $[0, +\infty)$.

(c) Prove that $\frac{x^{100}}{\sqrt{1-x}} = 2$ for some $x \in (0, 1)$.
(You may use without proof that $g(x) = x^{100}$ and $h(x) = 1 - x$ are continuous functions.)

Question 4 - Extra Credit

Let $0 \leq \alpha < 1$, and let $f : \mathbb{R} \to \mathbb{R}$ satisfy

$$|f(x) - f(y)| \leq \alpha|x - y|, \qquad \text{for all } x, y \in \mathbb{R}.$$

(Such a function is called an α-Lipschitz function. When $\alpha \in [0, 1)$, it is known as a contraction mapping.)

Let $a_1 \in \mathbb{R}$, and let $a_{n+1} = f(a_n)$ for $n \in \mathbb{N}$.

(a) Prove that a_n is a Cauchy sequence.

(b) Prove that there exists a unique real number $a \in \mathbb{R}$ so that $f(a) = a$.

In this problem, you have shown that any contraction mapping has a unique fixed point.