Math CCS 117: Practice Midterm 1

(Not to be turned in)

Question 1

Consider a nonempty set $A \subseteq \mathbb{R}$.

(a) Suppose A is bounded above. Prove that there exists a sequence a_n, satisfying $\{a_n : n \in \mathbb{N}\} \subseteq A$ and

$$\sup A - \frac{1}{n} \leq a_n \leq \sup A \text{ for all } n \in \mathbb{N}.$$

(b) Prove that the sequence you found in the previous part satisfies $\lim_{n\to\infty} a_n = \sup A$.

(c) Now suppose A is not bounded above. Prove that there exists a sequence a_n satisfying $\{a_n : n \in \mathbb{N}\} \subseteq A$ and

$$a_n \geq n \text{ for all } n \in \mathbb{N}.$$

(d) Prove that the sequence you found in the previous part satisfies $\lim_{n\to+\infty} a_n = \sup A$

In summary, you have proved the following important result: for any nonempty set $A \subseteq \mathbb{R}$, we may always find a sequence of elements a_n in A so that $\lim_{n\to+\infty} a_n = \sup A$.

Question 2

Suppose the limits of the sequences s_n and t_n exist and $a \in \mathbb{R}$.

(a) Suppose $\lim_{n\to\infty} s_n < a$. Prove that $s_n \geq a$ for at most finitely many n—in other words, prove that the set $\{n \in \mathbb{N} : s_n \geq a\}$ is finite.

(b) Suppose $\lim_{n\to+\infty} t_n > 0$. Prove that there exists $b > 0$ so that $t_n \geq b$ for all but finitely many n—in other words, prove that the set $\{n \in \mathbb{N} : t_n < b\}$ is finite.

Question 3

Consider a sequence a_n, and suppose that there exists $a \in \mathbb{R}$ so that

$$\lim_{n\to+\infty} a_{2n} = a = \lim_{n\to+\infty} a_{2n-1}.$$

Prove that $\lim_{n\to+\infty} a_n = a$.

Question 4 - Extra Credit

In this problem, you will show that the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$ is the limit of the following continued fraction:

$$1 + \frac{1}{1 + \frac{1}{1+\frac{1}{1+\dots}}}$$

Define a sequence s_n recursively as follows: $s_1 = 1$ and $s_{n+1} = 1 + \frac{1}{s_n}$. Prove that s_n converges to $\varphi = \frac{1+\sqrt{5}}{2}$. (Hint: one way to do this is to use the result of question 3.)

Additional Practice Problems

Question 5

Consider a sequence $a_n : \mathbb{N} \to \mathbb{R}$ satisfying $a_n \neq 0$, for all but finitely many $n \in \mathbb{N}$. If $\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = 0$, find $\lim_{n\to+\infty} a_n$.

Question 6

Let $(0, +\infty)$ denote the set $\{x \in \mathbb{R} : x > 0\}$.

(a) State the definition of what it means for a nonempty set $S \subseteq \mathbb{R}$ to be bounded below.

(b) Consider a nonempty set $S \subseteq (0, +\infty)$. Define $S' = \{1/s : s \in S\}$. Prove that $a > 0$ is a lower bound for S if and only if $1/a > 0$ is an upper bound for S'.

(c) Suppose $\inf S > 0$. Prove that $\sup S' = 1/\inf S$.

(d) Suppose $\inf S = 0$. Prove that $\sup S' = +\infty$.

Question 7

Which of the following is a correct version of the definition of convergence?

A sequence s_n converges to a limit s if...

(a) there exists $\epsilon > 0$ so that, for all $N \in \mathbb{R}$, $n > N$ ensures $|s_n - s| < \epsilon$.

(b) for all $\epsilon \geq 0$, there exists $N \in \mathbb{R}$ so that $n > N$ ensures $|s_n - s| < \epsilon$.

(c) for all $\epsilon > 0$, $|s_n - s| < \epsilon$ for all $n \in \mathbb{N}$.

(d) given $\epsilon > 0$, there is some $N \in \mathbb{R}$ so that $|s_n - s| < \epsilon$ for all $n > N$.

For each of the statements above that are *not* the correct definition of convergence, give an example of either

- a sequence that satisfies the statement but does not converge or
- a sequence that converges but does not satisfy the statement.

In each case, the existence of such an example will illustrate why the statement is *not* equivalent to the correct definition of convergence.