Practice Midterm 1 Solutions

CS 117, S25

1

a

Pf: For all $n \in \mathbb{N}$, $\sup(A) - \frac{1}{n} < \sup(A)$. Since $\sup(A)$ is the <u>least</u> upper bound of A, $\sup(A) - \frac{1}{n}$ cannot be an upper bound for A. That is, $\exists\ a_n \in A$ s.t. $\sup(A) - \frac{1}{n} \leq a_n$. Since $a_n \in A$, for all $n \in \mathbb{N}$, we have

$$\sup A - \frac{1}{n} \leq a_n \leq \sup A.$$

b

Fix $\varepsilon > 0$. By the Archimedean theorem, $\exists\ N \in \mathbb{R}$ s.t. $\frac{1}{N} < \varepsilon$. Thus, $n > N$ ensures $\frac{1}{n} < \frac{1}{N} < \varepsilon$, so

$$\sup A - \varepsilon < \sup A - \frac{1}{n} \leq a_n \leq \sup A$$

and $|a_n - \sup A| < \varepsilon$. This shows $\lim_{n \to \infty} a_n = \sup A$.

(c) Fix $n \in \mathbb{N}$. Since A is not bounded above, n is not an upper bound for A, so there exists $a_n \in A$ s.t. $a_n > n$. In this way, we know there exists a sequence a_n with $a_n \in A$ and $a_n > n$ for all $n \in \mathbb{N}$.

(d) Fix $M > 0$. Let $N = M$. Then for all $n > N$, $a_n > n > N = M$. Thus a_n diverges to $+\infty$. This shows $\lim_{n\to\infty} a_n = \sup A$

Thus, by definition of the greatest lower bound, $\frac{1}{m} \leq \inf S$. Thus, $\frac{1}{\inf S} \leq m$. This shows that $\frac{1}{\inf S}$ is the least upper bound of S'.
Thus $\frac{1}{\inf S} = \sup S'$.

(d) By definition of what $\sup S' = +\infty$ means, it suffices to show S' is not bounded above.

Suppose, for the sake of contradiction, that m were an upper bound for S'. Again, since $S' \subseteq (0, +\infty)$ is nonempty, we have $m > 0$. By part (b), $\frac{1}{m} > 0$ is a lower bound for S. This contradicts that $\inf S = 0$ is the greatest lower bound.

Thus, S' is not bounded above.

(2) (a) First, suppose $\lim_{n\to\infty} s_n = s$ for $s<a$. If we define $\varepsilon = a-s$, then $\varepsilon>0$. By definition of convergence, there exists $N\in\mathbb{R}$ s.t. $n>N$ ensures $|s_n-s|<\varepsilon \iff s-\varepsilon<s_n<s+\varepsilon$

Thus $n>N$ ensures $s_n<s+\varepsilon=a$. Hence $\{n\in\mathbb{N} : s_n\geq a\}\subseteq\{1,2,\dots,\lfloor N\rfloor\}$. Thus, $\{n\in\mathbb{N}: s_n\geq a\}$ is a finite set.

Next, suppose $\lim_{n\to\infty} s_n = -\infty$. Let $M=\min\{-1,a\}$. Then $\exists$ N s.t. $s_n<M\leq a$ for all $n>N$. Thus, $\{n\in\mathbb{N}: s_n\geq a\}$ is a finite set.

(b) First, suppose $\lim_{n\to\infty} t_n = t>0$. Then $\lim_{n\to\infty} t_n > \frac{t}{2}$. Applying part (a) with $s_n=-t_n$ and $a=\cancel{0}-\frac{t}{2}$, we obtain that

$$\{n\in\mathbb{N}: s_n\geq a\}=\{n\in\mathbb{N}: -t_n\geq -\tfrac{t}{2}\}$$
$$=\{n\in\mathbb{N}: t_n\leq \tfrac{t}{2}\}\supseteq\{n\in\mathbb{N}: t_n<\tfrac{t}{2}\}$$

are all finite sets. This shows the result for $b=\frac{t}{2}$.

Now, suppose $\lim_{n\to\infty} t_n = +\infty$. Then $\exists$ N s.t. $\forall n > N$, $t_n > 1$. Thus, $\{n \in \mathbb{N} : t_n \leq 1\} \supseteq \{n \in \mathbb{N} : t_n < 1\}$ is finite. This shows the result for $b = 1$.

(3)

Fix $\varepsilon > 0$. Since $\lim_{n\to\infty} a_{2n} = a = \lim_{n\to\infty} a_{2n+1}$, $\exists$ $N_e, N_o \in \mathbb{R}$ s.t.

$$n > N_e \Rightarrow |a_{2n} - a| < \varepsilon$$
$$n > N_o \Rightarrow |a_{2n+1} - a| < \varepsilon.$$

Define $N = 2 \cdot \max\{N_e, N_o\}$. Suppose $n > N$. If n is even, $n = 2m$ for $m \in \mathbb{N}$ and $n > N \geq 2N_e \Rightarrow 2m \geq 2N_e \Rightarrow m \geq N_e$, so $|a_n - a| = |a_{2m} - a| < \varepsilon$. OTOH, if n is odd, $n = 2m-1$ for $m \in \mathbb{N}$, and $n > N \geq 2N_o - 1 \Rightarrow 2m - 1 \geq 2N_o - 1 \Rightarrow m \geq N_o$, so $|a_n - a| = |a_{2m-1} - a| < \varepsilon$. Thus, in both cases, $n > N$ ensures $|a_n - a| < \varepsilon$. This shows $\lim_{n\to\infty} a_n = a$.

④

We begin by showing s_{2n} is decreasing and s_{2n-1} is increasing.

We proceed by induction. For the base case, note that $s_1 = 1$, $s_2 = 2$, $s_3 = \frac{3}{2}$, $s_4 = \frac{5}{3}$. Suppose $s_{2k} \geq s_{2(k+1)}$ and $s_{2k-1} \leq s_{2k+1}$.
Then $s_{2(k+2)} = 1 + \frac{1}{s_{2k+1}} \leq 1 + \frac{1}{s_{2k-1}} = s_{2k}$,
so $s_{2k+3} = 1 + \frac{1}{s_{2(k+2)}} \geq 1 + \frac{1}{s_{2k}} = s_{2k+1}$.

Now, we show $1 \leq s_n \leq 2$ $\forall n \in \mathbb{N}$.
Since s_{2n} is decreasing and $s_2 = 2$, we have $s_{2n} \leq 2$ $\forall n$. Since s_{2n-1} is increasing and $s_1 = 1$, we have $s_{2n-1} \geq 1$ $\forall n$. Furthermore,
$s_{2n} = 1 + \frac{1}{s_{2n-1}} \overset{s_{2n-1}>0}{\geq} 1$ $\forall n$ and
$s_{2n+1} = 1 + \frac{1}{s_{2n}} \overset{s_{2n}\geq 1}{\leq} 2$. This shows
$1 \leq s_n \leq 2$ $\forall n \in \mathbb{N}$.

Since $1 \leq s_n \leq 2$ for all n, the subsequence of even terms and the subsequence of odd terms are both bounded and monotone. Hence, they both converge. Let $\lim_{k\to\infty} S_{2k} = S_{even}$ and $\lim_{k\to\infty} S_{2k-1} = S_{odd}$.

Note that:

$$S_{2k+1} = 1 + \frac{1}{S_{2k}} = 1 + \frac{1}{1+\frac{1}{S_{2k-1}}}.$$

Since $S_{2k-1} \geq 1$, $S_{odd} \geq 1$. Thus, applying the limit theorems (quotient, sum), we have $S_{odd} = \lim_{k\to\infty} S_{2k+1} = 1 + \frac{1}{1+\frac{1}{S_{odd}}}$.

Thus, S_{odd} solves $(S_{odd}-1) = \left(1+\frac{1}{S_{odd}}\right)^{-1}$

$\overset{S_{odd}\geq 1}{\iff} (S_{odd}-1)\left(1+\frac{1}{S_{odd}}\right) = 1 \iff S_{odd} + 1 - 1 + \frac{1}{S_{odd}} = 1$

$\iff S_{odd}^2 - S_{odd} - 1 = 0$. By the quadratic formula and the fact that $S_{odd} \in [1,2]$, we obtain $S_{odd} = \varphi$.

Finally,

$$S_{2k} = 1 + \frac{1}{S_{2k-1}}.$$

Again, applying the limit theorems, we obtain $S_{even} = 1 + \frac{1}{S_{odd}} \Rightarrow S_{even} = \varphi$.

The result then follows by question 3.

(5) We will show $\lim_{n\to\infty} a_n = 0$. Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$, $\exists\, N_0$ s.t. $n > N_0$ ensures $\left|\frac{a_{n+1}}{a_n}\right| < \frac{1}{2} \Rightarrow \underbrace{|a_{n+1}| < \frac{1}{2}|a_n|}_{*}$.

We claim that, $\forall\, N, m \in \mathbb{N}$ with $N > N_0$, $\underbrace{|a_{N+m}| < (\tfrac{1}{2})^m |a_N|}_{**}$. We have just shown the base case $m=1$.
Suppose $|a_{N+m}| < (\frac{1}{2})^m |a_N|$. By $(*)$, $|a_{N+m+1}| < \frac{1}{2}|a_{N+m}| < (\frac{1}{2})^{m+1}|a_N|$.
This proves the claim.

Since $\frac{1}{2} \in (0,1)$, by HW3 Q5, $\lim_{m\to\infty} (\frac{1}{2})^m = 0$, so $\forall\, N \in \mathbb{N}$ with $N > N_0$, $\lim_{m\to\infty} (\frac{1}{2})^m |a_N| = 0$, by the theorem that the limit of the product is the product of the limits.

Fix $\varepsilon > 0$. Then $\exists M$ s.t. $m > M$ ensures $\left(\frac{1}{2}\right)^m |a_{N_0+1}| < \varepsilon$. Thus for $n > N_0 + M + 2$, we may express $n = N + m$ for $N > N_0 + 1$ and $m > M$, so $**$ ensures $|a_n| = |a_{N+m}| < \left(\frac{1}{2}\right)^m |a_{N_0+1}| < \varepsilon$.

Therefore, $\lim_{n \to \infty} a_n = 0$.

⑥

(a) S is bounded below if there exists $m_0 \in \mathbb{R}$ s.t. $m_0 \leq s$ for all $s \in S$.

(b) Suppose $a > 0$ is a lower bound for S. Then $a \leq s$ $\forall s \in S$, which is equivalent to $\frac{1}{s} \leq \frac{1}{a}$ $\forall s \in S$. Furthermore, this is equivalent to $t \leq \frac{1}{a}$ $\forall t \in S'$. Finally, we recognize this is equivalent to $\frac{1}{a}$ being an upper bound for S'.

(c) If $\inf S > 0$, then by part (b), we use the fact that $\inf S$ is a lower bound for S to conclude that $\frac{1}{\inf S}$ is an upper bound for S'.

Suppose M is an upper bound for S'. Since $S' \subseteq (0, +\infty)$ is nonempty, M must be strictly positive. By part (b), $\frac{1}{M}$ is a lower bound for S.

Thus, by definition of the greatest lower bound, $\frac{1}{m} \leq \inf S$. Thus, $\frac{1}{\inf S} \leq m$. This shows that $\frac{1}{\inf S}$ is the least upper bound of S'. Thus $\frac{1}{\inf S} = \sup S'$.

(d) By definition of what $\sup S' = +\infty$ means, it suffices to show S' is not bounded above.

Suppose, for the sake of contradiction, that m were an upper bound for S'. Again, since $S' \subseteq (0, +\infty)$ is nonempty, we have $m > 0$. By part (b), $\frac{1}{m} > 0$ is a lower bound for S. This contradicts that $\inf S = 0$ is the greatest lower bound.

Thus, S' is not bounded above.

(7) The correct definition is (d).

(a) Consider $s_n = (1,-1,1,-1,\dots)$, $s=0$. Then for $\varepsilon = 2$ and all $N \in \mathbb{R}$, $n > N$ ensures $|s_n - s| < \varepsilon$.

(b) Consider $s_n = \frac{1}{n}$, $s=0$. For $\varepsilon = 0$ there is no $N \in \mathbb{R}$ so that $n > N$ ensures $|s_n - 0| < \varepsilon$.

(c) Consider $s_n = \frac{1}{n}$, $s=0$. For $\varepsilon = \frac{1}{4}$, $|s_n - s| < \varepsilon$ is not true for all $n \in \mathbb{N}$.