Math CCS 117: Practice Midterm 1

(Not to be turned in)

Question 1

(a) Let s_n be a bounded sequence of real numbers. Let A be the set of $a \in \mathbb{R}$ such that $\{n \in \mathbb{N} : s_n < a\}$ is finite. In other words, A is the set of real numbers a for which at most finitely many s_n are less than a. Prove that $\sup A = \liminf s_n$.

(b) How would you want to define $\sup \emptyset$, where $\emptyset$ is the empty set, in order to make the result true for unbounded sequences s_n? You do not need to justify your answer.

Question 2

Define a sequence s_n as follows: $s_1 = 1$ and, for $n > 1$, $s_{n+1} = \left(\frac{n}{n+1}\right) s_n^2$.

(a) Prove that $0 \leq s_n \leq 1$ for all $n \in \mathbb{N}$.

(b) Prove that s_n is a decreasing sequence.

(c) Explain why s_n converges.

(d) Use the definition of s_n to find the value of s, where $s = \lim_{n \to +\infty} s_n$.

Question 3

In this question, we will justify our notation $a^{1/n}$ for $a \geq 0$ by proving that, for any $a \geq 0$, there exists a unique $x \geq 0$ so that $x^n = a$.

(a) For any $n \in \mathbb{N}$, prove that $f(x) = x^n$ is a continuous function.

(b) Use properties of continuous functions to prove that, for any $a \geq 0$, there exists $x \geq 0$ so that $f(x) = a$.

(c) Prove that the $x \geq 0$ found in part (ii) is unique.

Question 4 - Extra Credit

Let $0 \leq \alpha < 1$, and let f be a function from $\mathbb{R}$ to $\mathbb{R}$ that satisfies

$$|f(x) - f(y)| \leq \alpha |x - y|, \quad \text{for all } x, y \in \mathbb{R}.$$

(Such a function is called an α-Lipschitz function.)

Let $a_1 \in \mathbb{R}$, and let $a_{n+1} = f(a_n)$ for $n \in \mathbb{N}$. Prove that a_n is a Cauchy sequence.

Additional Practice Problems

Question 5

(a) Prove that, for any $c \in \mathbb{R}$, the constant function $f(x) = c$ is continuous. Prove that, for any $k \in \mathbb{R}$, the function $g(x) = kx$ is continuous.

(b) Consider the function

$$f(x) = \begin{cases} 1/x & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

Prove that $f(x)$ is not continuous.

Question 6

Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. Prove that f is continuous at x_0 in $\text{dom}(f)$ if and only if for every sequence x_n in $\text{dom}(f) \setminus \{x_0\}$ that converges to x_0, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.

(Hint: To show that if f satisfies the above criteria then it is continuous at x_0, proceed by contradiction. Using HW6, Q2(c), explain why there exists $\epsilon > 0$ and a subsequence x_{n_k} so that $|f(x_{n_k}) - f(x_0)| \geq \epsilon \ \forall k \in \mathbb{N}$, while $\lim_{k\to+\infty} x_{n_k} = x_0$. Explain why this is a contradiction.)