Practice Midterm 2 Solutions, CS117, S25
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(1)

First, we show $\liminf_{n\to\infty} s_n$ is an upper bound for A.
Fix $a \in A$. Then $\exists\ N_a$ s.t. $n \geq N_a$ ensures $s_n \geq a$. Thus, for $N > N_b$, $b_N := \inf\{s_n : n > N\} \geq a$.
Thus,

$$\liminf_{n\to\infty} s_n = \lim_{N\to\infty} \inf\{s_n : n > N\}$$
$$\geq a$$

Our argument above shows that $\liminf_{n\to\infty} s_n \geq \sup(A)$.
Suppose, for the sake of contradiction, that $\sup(A) < \liminf_{n\to\infty} s_n$. Then $\exists\ r \in \mathbb{R}$ s.t. $\sup(A) < r < \liminf_{n\to\infty} s_n$. Since $r \notin A$, $|\{n \in \mathbb{N} : s_n < r\}|$ is infinite. Thus, $\inf\{s_n : n > N\} \leq r\ \forall N \in \mathbb{N}$.

This implies $\liminf_{n\to\infty} s_n \leq r$, which is a contradiction.

(b) $\sup \emptyset = -\infty$

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(a) We proceed by induction. It is clear that $0 \le s_1 \le 1$. Suppose $0 \le s_n \le 1$. then since $\frac{n}{n+1} \ge 0$ $s_{n+1} = \left(\frac{n}{n+1}\right) s_n^2 \ge 0$. Likewise, since $\frac{n}{n+1} \le 1$ and $s_n^2 \le s_n \le 1$, $\left(\frac{n}{n+1}\right) s_n^2 \le 1$. This gives the result.

(b) Since $0 \le s_n \le 1$, $s_n^2 \le s_n$. Since $0 \le \frac{n}{n+1} \le 1$, $s_{n+1} = \left(\frac{n}{n+1}\right) s_n^2 \le 1 \cdot s_n^2 \le s_n$. This shows it is decreasing.

(c) All bounded monotone sequences converge.

(d) Let $s \in \mathbb{R}$ be the limit of s_n. Since s_{n+1} has the same limit,

$$s = \lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right) s_n^2.$$

Since s_n is convergent, $\lim_{n\to\infty} s_n^2 = (\lim_{n\to\infty} s_n)(\lim_{n\to\infty} s_n) = s^2$.
Since $\frac{n}{n+1} = \frac{1}{1+\frac{1}{n}}$ converges to 1, $\lim_{n\to\infty} \left(\frac{n}{n+1}\right) s_n^2 = s^2$.
Thus $s = s^2$. We must have either $s=0$ or $s=1$. Since s_n is decreasing and $s_2 = \frac{2}{3}$, we have $s=0$.

(3)

(a) Fix arbitrary $x_0 \in \text{dom}(f) = \mathbb{R}$. Fix an arbitrary sequence x_k converging to x_0. Assume $x_k^\ell \to x_0^\ell$ for $\ell \in \mathbb{N}$. Since the limit of the product is the product of the limits, $x_k^{\ell+1} = x_k^\ell x_k \to x_0^\ell x_0 = x_0^{\ell+1}$! Thus, by induction, $x_k^n \to x_0^n$ $\forall n \in \mathbb{N}$. This shows f is continuous.

(b) The result is clearly true for $a=0$, so suppose $a>0$. Note that, if $a \leq 1$, $a^n \leq 1$ $\forall n \in \mathbb{N}$. Likewise, if $a > 1$, $a < a^n$ $\forall n \in \mathbb{N}$. Define

$$a_* = \begin{cases} 1 & \text{if } a \leq 1 \\ a & \text{if } a > 1. \end{cases}$$

Then $a_* > 0$ and $f(a_*) \geq a$.

Since f is cts, $f(0)=0$, and $f(a_*) \geq a$, the Intermediate Value Theorem applied to the closed interval $[0, a_*]$ ensures there exists $x \in [0, a_*]$ so that $f(x) = a \in [f(0), f(a_*)]$.

(c) Assume, for the sake of contradiction, that $\exists\, x, y \geq 0$ s.t. $x \neq y$ and $x^n = y^n = a$. WLOG, suppose $x < y$.

Assume $x^{\ell} < y^{\ell}$ for some $\ell \in \mathbb{N}$. Then $x^{\ell+1} < x y^{\ell} < y y^{\ell} = y^{\ell+1}$. Thus $x^n < y^n$ $\forall n \in \mathbb{N}$. This contradicts that $x^n = y^n = a$.

(4) :)

(5) (a) Fix $x_0 \in \mathbb{R}$ and a sequence x_n converging to x_0. We must show $f(x_n)$ converges to $f(x_0)$. Since $f(x_n) = c$ for all $n \in \mathbb{N}$ and $f(x_0) = c$, this is trivially true.

(b) Take $x_0 = 0$ and $x_n = \frac{1}{n}$. Then $\lim_{n \to \infty} x_n = x_0$ but $\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} n = +\infty$ $\neq 0 = f(x_0)$.

(4) Suppose f is cts at x_0. By defn of cty, for every sequence $x_n \in dom(f)$ s.t. $\lim_{n\to\infty} x_n = x_0$ we have $\lim_{n\to\infty} f(x_n) = f(x_0)$. Since $(dom(f) \setminus \{x_0\}) \subseteq dom(f)$, for every sequence $x_n \in dom(f) \setminus \{x_0\}$ s.t. $\lim_{n\to\infty} x_n = x_0$ we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.

(*) Now, suppose that for every sequence $x_n \in dom(f) \setminus \{x_0\}$ s.t. $\lim_{n\to\infty} x_n = x_0$ we have $\lim_{n\to\infty} f(x_n) = f(x_0)$. Let y_n be an arbitrary sequence in $dom(f)$ s.t. $\lim_{n\to\infty} y_n = x_0$.

We must show $\lim_{n\to\infty} f(y_n) = f(x_0)$. Assume, for the sake of contradiction, that $f(y_n)$ does not converge to $f(x_0)$.

By HW6, Q2(c), $\exists\, \varepsilon > 0$ and a subsequence $f(y_{n_k})$ such that $|f(y_{n_k}) - f(x_0)| \geq \varepsilon \;\; \forall\, k \in \mathbb{N}$. (**) Note that this is only possible if $y_{n_k} \neq x_0$ for all $k \in \mathbb{N}$. However this means $y_{n_k} \in \text{dom}(f) \setminus \{x_0\}$, and since y_{n_k} is a subsequence of the convergent sequence y_n, $\lim_{k \to \infty} y_{n_k} = x_0$

By assumption (*), this implies $\lim_{k \to \infty} f(y_{n_k}) = f(x_0)$. This contradicts (**).