Math 117: Homework 3

Due Thursday, April 23rd at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1

Given a sequence s_n, suppose that $s, \tilde{s} \in \mathbb{R}$ are both limits of the sequence. Prove that $s = \tilde{s}$.

This shows the limit of a sequence is unique and justifies the fact that we refer to "the" limit of a sequence.

Question 2

Consider $x, y \in \mathbb{R}$ satisfying $x, y \in [1, 2]$. Suppose $x^2 < 2$ and $y^2 > 2$.

(a) Suppose $0 < \epsilon < 1$. Prove that $(x + \epsilon)^2 \leq x^2 + 5\epsilon$ and $(y - \epsilon)^2 \geq y^2 - 4\epsilon$.

(b) Prove that there exists $\epsilon_1, \epsilon_2 \in (0, 1)$ so that $x^2 + 5\epsilon_1 < 2$ and $y^2 - 4\epsilon_2 > 2$.

(c) Use parts (a) and (b) to show that there exists $\epsilon_1, \epsilon_2 \in (0, 1)$ so that $(x + \epsilon_1)^2 < 2$ and $(y - \epsilon_2)^2 > 2$.

Question 3*

Consider an ordered field F. For any $a \in F$, recall that a^2 is an abbreviation for $a \cdot a$.

Fix $a \in F$ with $a \geq 0$. Consider the set $S = \{c \in F : c \geq 0,\ c^2 \leq a\}$

(a) Prove that S is nonempty and bounded above.

(b) Suppose $F = \mathbb{R}$. Explain why, in this case, the supremum of S exists.

Question 4*

Consider the set $S = \{c \in \mathbb{R} : c \geq 0,\ c^2 \leq 2\}$. Let $b = \sup(S)$.

(a) Prove that $b \in [1, 2]$.

(b) Prove that $b^2 \geq 2$. (Hint: proceed by contradiction, using question 2).

(c) Prove that $b^2 \leq 2$. (Hint: proceed by contradiction, using question 2).

Combining parts (b) and (c), we see that $b^2 = 2$.

In this way, we have shown there exists a real number $b > 0$ so that $b^2 = 2$. We can now define the symbol $\sqrt{2}$ by setting $\sqrt{2} := b$. In this way, we have proved $\sqrt{2} \in \mathbb{R}$. Combining this with our result from class that $\sqrt{2} \notin \mathbb{Q}$, we see that $\sqrt{2} \in \mathbb{I}$, where $\mathbb{I} := \mathbb{R} \setminus \mathbb{Q}$ is the set if *irrational numbers*.

Question 5*

(a) Prove the following, using the definition of convergence:

$$\lim_{n\to+\infty} a^n = \begin{cases} 0 & \text{if } |a|<1. \\ 1 & \text{if } a=1. \end{cases}$$

You may use standard facts about the natural logarithm on the real numbers, even though we haven't proved them yet. In particular, you may use that the natural logarithm $\log(x)$ is an increasing function for $x>0$: that is, for any $x,y>0$, $x\le y \iff \log(x)\le\log(y)$.

(b) If $a\le -1$ prove that the sequence does not converge.

Question 6

Fix $a\in\mathbb{R}$ and consider the collection of rational numbers $S=\{q\in\mathbb{Q}: a\le q\}$.

(a) Suppose the underlying field is either $\mathbb{F}=\mathbb{R}$ or $\mathbb{F}=\mathbb{Q}$. For which values of a does the minimum of S exist? Justify your answer with a proof.

(b) Suppose the underlying field is $\mathbb{F}=\mathbb{R}$. Prove that $\inf(S)=a$.

Question 7

Given $s,t\in\mathbb{R}$, consider the set $(s,t]=\{x\in\mathbb{R}: s<x\le t\}$. Find the maximum, minimum, supremum, and infimum of the set or state that they do not exist. Justify your answers with proofs.

Question 8

Consider a sequence s_n. Prove that s_n is a bounded sequence if and only if $S:=\{s_n : n\in\mathbb{N}\}$ is a bounded set.

Question 9*

(a) State the definition of what it means for a sequence s_n to converge to a limit s.

(b) State the definition of what it means for a sequence s_n to *not* converge to a limit s, by negating the definition of convergence.

(c) Use the definition of a convergent sequence to prove that $\lim_{n\to+\infty}\frac{n-3}{n^2+9}=0$.

(d) Use the definition of a convergent sequence to prove that the sequence $s_n=(n+1)^2-2$ does not converge.

Question 10*

Determine if the following sequences converge. Justify your answer with a proof.

(a) $a_n=\frac{7n-19}{3n+7}$

(b) $b_n=\sin\left(\frac{n\pi}{3}\right)$

You may use standard facts about trigonometric functions, even though we haven't proved them.

Question 11*

Consider two sequences, a_n and b_n. Suppose a_n converges to a and that there exists $N \in \mathbb{R}$ so that $b_n = a_n$ for all $n \geq N$. Prove that b_n converges to a.

Question 12*

Which of the following is a correct version of the definition of convergence?

A sequence s_n converges to a limit s if...

(a) there exists $\epsilon > 0$ so that, for all $N \in \mathbb{R}$, $n > N$ ensures $|s_n - s| < \epsilon$.

(b) for all $\epsilon \geq 0$, there exists $N \in \mathbb{R}$ so that $n > N$ ensures $|s_n - s| < \epsilon$.

(c) for all $\epsilon > 0$, $|s_n - s| < \epsilon$ for all $n \in \mathbb{N}$.

(d) given $\epsilon > 0$, there is some $N \in \mathbb{R}$ so that $|s_n - s| < \epsilon$ for all $n > N$.

For each of the statements above that are *not* the correct definition of convergence, give an example of either

- a sequence that satisfies the statement but does not converge or
- a sequence that converges but does not satisfy the statement.

In each case, the existence of such an example will illustrate why the statement is *not* equivalent to the correct definition of convergence.