Math 117: Homework 4

Due Monday, May 5th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1

(a) Suppose $\lim_{n\to+\infty} s_n = +\infty$ and $\lim_{n\to+\infty} t_n > 0$. Prove that $\lim_{n\to+\infty} s_n t_n = +\infty$.

(b) If either limit the limit of t_n or $-t_n$ exists, prove that $\lim_{n\to+\infty} -t_n = -\lim_{n\to+\infty} t_n$.

Question 2

(a) Suppose t_n converges to a limit t. Prove that $\lim_{n\to+\infty} |t_n| = |t|$.

(b) In part (a) you proved that if t_n is a convergent sequence, then $|t_n|$ is a convergent sequence. Is the converse true? If so, prove it. If not, give a counterexample and justify your counterexample.

Question 3*

Prove the following theorem:

THEOREM 1. *If s_n and t_n are convergent sequences and $\lim_{n\to+\infty} s_n \neq 0$, then*

$$\lim_{n\to+\infty} \left(\frac{t_n}{s_n}\right) = \frac{\lim_{n\to+\infty} t_n}{\lim_{n\to+\infty} s_n}.$$

Question 4*

Prove the following useful lemma:

LEMMA 2. *If r_n and t_n are sequences whose limits exist and $r_n \leq t_n \ \forall n \in \mathbb{N}$, then*

$$\lim_{n\to+\infty} r_n \leq \lim_{n\to+\infty} t_n.$$

Do **not** assume that the sequences r_n and t_n converge, only that the limits exist.

Question 5*

An important lemma in the analysis of sequences is known as the *squeeze lemma.*

LEMMA 3 (Squeeze Lemma)**.** *Consider three sequences a_n, b_n, and s_n. If $a_n \leq s_n \leq b_n$ for all $n \in \mathbb{N}$ and*

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} b_n,$$

then $\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} a_n = \lim_{n\to+\infty} b_n.$.

(a) Prove the squeeze lemma. (Hint: use Q4.)

(b) Suppose s_n and t_n are sequences satisfying $|s_n| \leq t_n$ for all $n \in \mathbb{N}$. If $\lim_{n\to+\infty} t_n = 0$, prove that $\lim_{n\to+\infty} s_n = 0$.

(c) Is the converse to part (b) true? If so, prove it. If not, give a counterexample and justify your counterexample.

Question 6*

Suppose the limits of the sequences s_n and t_n exist and $a \in \mathbb{R}$.

(a) Suppose $\lim_{n\to\infty} s_n < a$. Prove that $s_n \geq a$ for at most finitely many n—in other words, prove that the set $\{n \in \mathbb{N} : s_n \geq a\}$ is finite.

(b) Suppose $\lim_{n\to+\infty} t_n > 0$. Prove that there exists $b > 0$ so that $t_n \geq b$ for all but finitely many n—in other words, prove that the set $\{n \in \mathbb{N} : t_n < b\}$ is finite.

Question 7*

Prove $\limsup_{n\to+\infty} |s_n| = 0$ if and only if $\lim_{n\to+\infty} s_n = 0$.

Question 8

Determine whether the following statements are true or false. If they are true, prove them. If they are false, give a counterexample and justify it.

(a) If a sequence s_n satisfies $\limsup_{n\to+\infty} s_n = 2$, then $s_n > 1.99$ for all n large enough.

(b) If a sequence s_n satisfies $\limsup_{n\to+\infty} s_n = b$, then $s_n \leq b$ for all n large enough.

Question 9*

For any $a \in \mathbb{R}$, $a > 0$, prove that $\lim_{n\to+\infty} a^{1/n} = 1$.

Hint: One approach is to first prove the result for $a \geq 1$. In this case, begin by explaining why $a^{1/n}$ must converge to some $L \geq 1$. Then argue that $a^{2/n}$ must converge to $L^2 \geq 1$.

Question 10

Determine if the following statement is true or false. If it is true, prove it. If it is false, provide a counterexample.

If s_n and t_n are sequences whose limits exist and for which $s_n < t_n$ for all but finitely many $n \in \mathbb{N}$, then $\lim_{n\to+\infty} s_n < \lim_{n\to+\infty} t_n$.

Question 11*

(i) Prove the nested interval theorem: If $\{[a_n, b_n]\}_{n\in\mathbb{N}}$ is a collection of closed intervals satisfying

$$[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n], \text{ for all } n \in \mathbb{N},$$

then $\cap_{n\in\mathbb{N}}[a_n, b_n]$ is nonempty.

(ii) Find a necessary and sufficient condition that$\cap_{n\in\mathbb{N}}[a_n, b_n]$ contains a single point. Justify your answer with a proof.

Question 12

Given a sequence s_n, prove that $\limsup_{n\to+\infty} |s_n| < +\infty$ if and only if s_n is bounded.