Homework 4 Solutions, CCS 117, S25

Question 3

Since $\lim_{n\to\infty} s_n = s \neq 0$, $\exists\, N$ s.t. $n > N$ ensures $|s_n - s| < \frac{|s|}{2}$. By the reverse triangle inequality, this shows that $n>N$ ensures $|s| - |s_n| < \frac{|s|}{2} \iff \underbrace{\frac{|s|}{2} < |s_n|}_{(*)}$.

First, we will show $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$. Note that $\frac{1}{s_n}$ is well-defined for all $n>N$ by $(*)$. Fix $\varepsilon > 0$. Note that, for $n > N$,

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s - s_n}{s_n s}\right| = \frac{|s-s_n|}{|s_n||s|} \overset{(*)}{<} \frac{|s-s_n|\,2}{|s|^2}.$$

Since $s_n \to s$, $\exists\, \tilde{N}$ s.t. $n > \tilde{N}$ ensures $|s - s_n| < \frac{\varepsilon |s|^2}{2}$. Thus $n > \max\{N, \tilde{N}\}$ ensures $\left|\frac{1}{s_n} - \frac{1}{s}\right| < \varepsilon$. This shows $\frac{1}{s_n} \to \frac{1}{s}$.

The result of Q3 then follows from the fact that the limit of the product of convergent sequences is the product of the limits.

Question 4

(10) If $\lim_{n\to\infty} r_n = -\infty$, the result is immediate. Thus, it remains to consider the remaining cases.

Case 1: Suppose $\lim_{n\to\infty} r_n = r \in \mathbb{R}$

Case 1a: If $\lim_{n\to\infty} t_n = +\infty$, we are done.

Case 1b: Suppose $\lim_{n\to\infty} t_n = t \in \mathbb{R}$. Assume for the sake of contradiction that $t < r$. Let $\varepsilon = \frac{r-t}{2} > 0$. Then $\exists\, N_r, N_t$ s.t. $n > N_r$ ensures $|r_n - r| < \varepsilon$ and $n > N_t$ ensures $|t_n - t| < \varepsilon$. Let $N = \max\{N_r, N_t\}$. Then $n > N$ ensures

$$t_n < t + \varepsilon = t + \frac{r-t}{2} = \frac{t+r}{2} = r - \frac{r-t}{2} = r - \varepsilon < r_n.$$

This contradicts that $r_n \le t_n \ \forall n \in \mathbb{N}$. Thus $\lim_{n\to\infty} t_n = t \ge r = \lim_{n\to\infty} r_n$.

Case 1c: Suppose $\lim_{n\to\infty} t_n = -\infty$. Then $\exists\, N_t$ s.t. $\forall n > N_t$, $t_n < r - 1$. There also exists N_r s.t. $\forall n > N_t$, $r - 1 < r_n$. Thus for $N = \max\{N_t, N_r\}$, $n > N$ ensures

$$t_n < r - 1 < r_n.$$

Again, this contradicts that $r_n \le t_n \ \forall n \in \mathbb{N}$. Thus $\lim_{n\to\infty} t_n = -\infty$ is impossible.

Case 2: Suppose $\lim_{n\to\infty} r_n = +\infty$. Fix $M > 0$. Then $\exists\, N$ s.t. $\forall n \ge N$, $M < r_n \le t_n$. This shows $\lim_{n\to\infty} t_n = +\infty$.

Question 5

ⓐ Let $s := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

If $s = +\infty$, then $\forall M > 0$, $\exists N$ s.t. $n > N$ ensures $M < a_n \leq s_n$, so $\lim_{n\to\infty} s_n = +\infty$.

If $s = -\infty$, then $\forall M < 0$, $\exists N$ s.t. $n > N$ ensures $s_n \leq b_n < M$, so $\lim_{n\to\infty} s_n = -\infty$.

If $s \in \mathbb{R}$, then $\forall \varepsilon > 0$, $\exists N_a, N_b$ s.t. $n > N_a$ ensures $|a_n - s| < \varepsilon$ and $n > N_b$ ensures $|b_n - s| < \varepsilon$. Thus, $n > \max\{N_a, N_b\}$ ensures $s - \varepsilon < a_n \leq s_n \leq b_n < s + \varepsilon$, so $\lim_{n\to\infty} s_n = s$.

ⓑ Since $-t_n \leq s_n \leq t_n$ and $\lim_{n\to\infty} t_n = 0$ ensures $\lim_{n\to\infty} -t_n = 0$, the result is a consequence of the Squeeze Lemma.

ⓒ No. Consider the sequences $s_n = (0, 0, 0, \ldots)$ and $t_n = (1, 1, 1, \ldots)$.

Question 9

First suppose $r \geq 1$. Then, $\forall n \in \mathbb{N}$, $r^{n+1} = r^n r \geq r^n \cdot 1 = r$. Likewise, since $(\sqrt[n]{r})^n = \underbrace{(\sqrt[n]{r}) \cdot \ldots \cdot (\sqrt[n]{r})}_{n \text{ times}} = 1$,

we must have $\sqrt[n]{r} \geq 1$. Likewise,

$$\left(\sqrt[n+1]{r}\right)^{n+1} = r \leq \left(\sqrt[n]{r}\right)^n \sqrt[n]{r} = \left(\sqrt[n]{r}\right)^{n+1},$$

so $\sqrt[n+1]{r} \leq \sqrt[n]{r}$.

This shows that, when $a \geq 1$, $a^{1/n}$ is a decreasing sequence bounded below by 1, thus, it must converge to $L \geq 1$. Since the limit of the product is the product of the limits, $a^{2/n}$ must converge to $L^2 \geq 1$.

the even elements of the sequence $a^{2/n}$ must converge to the same limit.

since the even elements of the sequence are a subsequence of s_n.

However, the even elements of the sequence $a^{2/n}$ are exactly the elements of the sequence $(a^{1/n})$. This shows $L^2 = L (\geq 1$, so $L = 1$.

Now, suppose $a \in (0,1)$. Then $\frac{1}{a} \geq 1$, so $a^{-1/n} \to 1$. Since the limit of the quotient of convergent sequences (with nonzero denominator) is the quotient of the limit, $a^{1/n} \to 1$.