Math 117: Homework 5

Due Thursday, May 7th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1*

Consider the sequences defined as follows:

$$a_n = (-1)^{n+1}, \quad b_n = -\frac{1}{n}, \quad c_n = 2n.$$

(a) For each sequence, give its set of subsequential limits. Justify your answer.

(b) For each sequence, give its lim inf and lim sup. Justify your answer.

Question 2

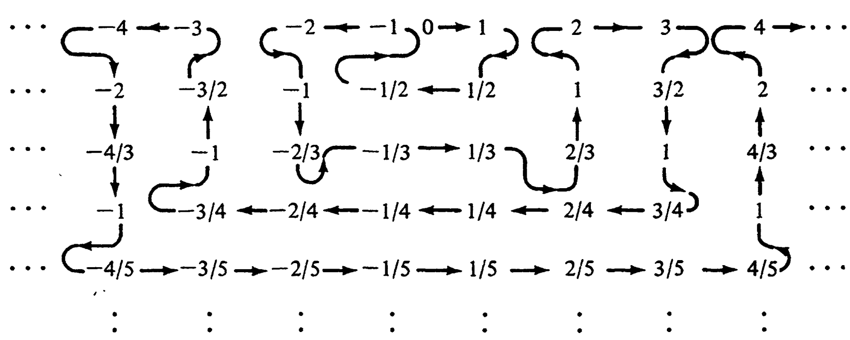
(a) State the definition of convergence for a sequence s_n to a limit s.

(b) State what it means for a sequence s_n to *not* converge to a limit s by negating the definition from part (a).

(c) Suppose that s_n does *not* converge to $s \in \mathbb{R}$. Prove that there exists $\epsilon > 0$ and a subsequence s_{n_k} so that $|s_{n_k} - s| \geq \epsilon$ for all k.

Question 3*

One can show that the set of rational numbers $\mathbb{Q}$ can be listed as a sequence r_n. The exact procedure is a little tedious, but you can get an idea of how it works by considering the below diagram from the textbook. For example, $r_1 = 0, r_2 = 1, r_3 = 1/2$, and so on. Note that some numbers, such as



-1, are included multiples times.

(a) For any $\epsilon > 0$ and $a \in \mathbb{R}$, show that the set $\{r \in \mathbb{Q} : |r - a| < \epsilon\}$ contains infinitely many elements.

(b) Let r_n be the sequence of rational numbers. Use part (a) to show that for any $a \in \mathbb{R}$, there exists a subsequence r_{n_k} that converges to a.

(c) Let r_n be the sequence of rational numbers. Show that there exists a subsequence r_{n_k} satisfying $\lim_{k \to +\infty} r_{n_k} = +\infty$.

Background on Infinite Series

In calculus, you encountered infinite series of the form

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \dots.$$

In fact, these are just limits of sequences. In particular, if we define the sequence

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \dots + a_n$$

to be the sum of the first n terms of the series, then we define the value of the infinite series to be

$$\sum_{k=1}^{\infty} a_k = \lim_{n \to +\infty} s_n.$$

DEFINITION 1. Given a series $\sum_{k=1}^{\infty} a_k$, define the sequence $s_n = \sum_{k=1}^{n} a_k$. Then the series $\sum_{k=1}^{\infty} a_k$ *converges* to a number L if and only if the sequence s_n converges to L. Likewise, the series *diverges to* $+\infty$ *or* $-\infty$ if and only if the sequence s_n diverges to $+\infty$ or $-\infty$.

Question 4* (Cauchy criterion)

Recall that a sequence s_n is a *Cauchy sequence* if

$$\text{for all } \epsilon > 0, \text{ there exists } N \in \mathbb{R} \text{ so that } n, m > N \text{ ensures } |s_n - s_m| < \epsilon.$$

(a) Prove that the following is an equivalent definition of a Cauchy sequence:

s_n is a Cauchy sequence if, for all $\epsilon > 0$, there exists $N \in \mathbb{R}$ so that $n > m > N$ ensures $|s_n - s_m| < \epsilon$.

(b) Prove the following theorem about series, known as the Cauchy criterion.

THEOREM 1 (Cauchy Criterion)**.** *A series $\sum_{k=1}^{\infty} a_k$ is convergent if and only if*

$$\textit{for all } \epsilon > 0 \textit{ there exists } N \in \mathbb{R} \textit{ so that } n > m > N \textit{ ensures } \left| \sum_{k=m+1}^{n} a_k \right| < \epsilon.$$

(c) Now use Theorem 1 to prove the following corollary:

COROLLARY 2. *If a series $\sum_{k=1}^{\infty} a_k$ is convergent, then $\lim_{k \to +\infty} a_k = 0$.*

Question 5

(a) Prove the following by induction: for $a \neq 1$,

$$\sum_{i=0}^{m-1} a^i = 1 + a + a^2 + \cdots + a^{m-1} = \frac{1-a^m}{1-a}.$$

(b) Use part (a) to show that

$$\sum_{i=n}^{m-1} a^i = a^n + a^{n+1} + \cdots + a^{m-2} + a^{m-1} = \frac{a^n - a^m}{1-a}.$$

(c) Recall that, by the triangle inequality,

$$\left| \sum_{i=1}^{n} a_i \right| = |a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n| = \sum_{i=1}^{n} |a_i|.$$

Let s_n be a sequence such that $|s_{n+1} - s_n| \leq 4^{-n}$ for all $n \in \mathbb{N}$. Use part (b) and the above inequality to prove s_n is a Cauchy sequence.

(d) Does the sequence from part (c) converge? Justify your answer.

Question 6* (decimal expansions)

In this problem you will show that any number that can be represented as a nonnegative decimal expansion can be thought of as the limit of a bounded increasing sequence of real numbers. Since all bounded monotone sequences converge, this guarantees that any decimal expansion you can imagine represents (converges to) a real number.

Suppose we are given a decimal expansion $K.d_1d_2d_3d_4\ldots$, where K is a nonnegative integer and each $d_j \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let

$$s_n = K + \frac{d_1}{10^1} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}.$$

(a) Show s_n is an increasing sequence. (This is almost obvious. Your proof should be short.)

(b) Use the result from Q5(a) to prove that $\frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} = 1 - \frac{1}{10^n}$.

(c) Use part (b) to prove that s_n is a bounded sequence.

(d) Since $0.\bar{9} = 0.999\ldots$ and 1 are both decimal expansions, by what you have shown, they both correspond to a real number. Use the hint from part (b) to show that they actually correspond to the same real number.

Question 7 (geometric series)

(a) Prove that for $|r| < 1$, $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$.

(b) Prove that for $|r| \geq 1$, $\sum_{k=0}^{\infty} r^k$ does not converge. (**Hint**: Use Corollary 2 from Q4.)

Question 8*

Let s_n be a sequence of nonnegative numbers, and for each n define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$.

(a) Show $\liminf s_n \le \liminf \sigma_n \le \limsup \sigma_n \le \limsup s_n$.
(**Hint:** For the first inequality, show that $M > N$ implies

$$\inf\{\sigma_n : n > M\} \ge \left(1 - \frac{N}{M}\right) \inf\{s_n : n > N\}.$$

For the last inequality, show first that $M > N$ implies

$$\sup\{\sigma_n : n > M\} \le \frac{1}{M}(s_1 + s_2 + \cdots + s_N) + \sup\{s_n : n > N\}.)$$

(b) Show that if $\lim s_n$ exists, then $\lim \sigma_n$ exists and $\lim \sigma_n = \lim s_n$.

(c) Give an example for which $\lim \sigma_n$ exists but $\lim s_n$ does not exist.

Question 9

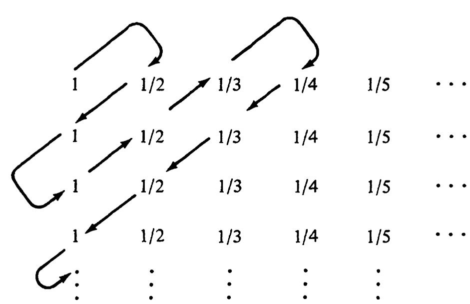
Suppose s_n and t_n are bounded sequences.

(a) Prove that $\limsup s_n + t_n \le \limsup s_n + \limsup t_n$.

(b) Give an examples of bounded sequences s_n and t_n for which $\limsup s_n + t_n < \limsup s_n + \limsup t_n$.

Question 10*

Let s_n be the sequence defined in the following figure from the textbook:



(a) Find the set S of subsequential limits of s_n. Justify your answer.

(b) Determine $\limsup s_n$ and $\liminf s_n$. Justify your answer.

Question 11

(a) Suppose s_n has a subsequence s_{n_k} that is bounded. Show that this implies s_n has a convergent subsequence.

(b) Suppose that s_n has no convergent subsequences. Prove that $\lim_{n\to+\infty} |s_n| = +\infty$.
(Hint: prove the result by contradiction.)

Question 12*

In this problem, we will consider sequences s_n satisfying the following property:

$$\exists s \in \mathbb{R} \text{ s.t. every subsequence } s_{n_k} \text{ of } s_n \text{ has a further subsequence } s_{n_{k_l}} \text{ satisfying } \lim_{l \to +\infty} s_{n_{k_l}} = s. \quad (*)$$

(a) Prove that if $\lim s_n = s$, then property (*) holds.

(b) Prove that if property (*) holds, then $\lim s_n = s$. (Hint: Use Q2, part c)