Math 117: Homework 5 Solutions

Question 1*

(a) For a_n, 1 and -1 are clearly subsequential limits, since the constant sequences $(1, 1, 1, \dots)$ and $(-1, -1, -1, \dots)$ are subsequences. Fix $t \in \mathbb{R} \setminus \{-1, 1\}$. Let $\epsilon = \min\{|t-1|, |t-(-1)|\}$. Then $\epsilon > 0$ and $\{n : |(-1)^n - t| < \epsilon\} = \emptyset$. By the main subsequences theorem, this implies that t is not a subsequential limit. Thus $\{-1, 1\}$ is the set of subsequential limits.

For $b_n = -1/n$, we have $\lim_{n\to\infty} b_n = 0$. Since every subsequence of a convergent sequence converges to the same limit, the set of subsequential limits is $\{0\}$.

For $c_n = 2n$, we have $\lim_{n\to\infty} c_n = +\infty$, so as in the previous part, the limit of every every subsequence is also $+\infty$.

(b) The lim inf and lim sup are the largest and smallest subsequential limits. Thus, $\liminf_{n\to\infty} a_n = -1$ and $\limsup_{n\to\infty} a_n = 1$. Likewise, $\liminf_{n\to\infty} b_n = \limsup_{n\to\infty} b_n = 0$ and $\liminf_{n\to\infty} c_n = \limsup_{n\to\infty} c_n = +\infty$.

(3) (a) We must show that for all $\varepsilon>0$ and $a\in\mathbb{R}$, $S=\{r\in\mathbb{Q}: a-\varepsilon<r<a+\varepsilon\}$ is infinite. We proceed by induction.
By denseness of $\mathbb{Q}$ in $\mathbb{R}$, there exists $r_1\in\mathbb{Q}$ so that $a-\varepsilon<r_1<a+\varepsilon$, so $r_1\in S$.
By denseness of $\mathbb{Q}$ in $\mathbb{R}$, there exists $r_2\in\mathbb{Q}$ so that $a-\varepsilon<r_2<r_1<a+\varepsilon$, so $r_2\in S$.
Assume we have picked k distinct elements $r_1,r_2,\dots,r_k\in S$ satisfying $r_k<r_{k-1}<\dots<r_2<r_1$.

By denseness of $\mathbb{Q}$ in $\mathbb{R}$, there exists $r_{k+1}\in\mathbb{Q}$ so that $a-\varepsilon<r_{k+1}<r_k<\dots<r_1<a+\varepsilon$, so $r_{k+1}\in S$. Thus S has infinitely many elements.

(b) Since $\{r\in\mathbb{Q}: |r-a|<\varepsilon\}$ contains infinitely many elements and r_n is the sequence of rational numbers, $\{n\in\mathbb{N}: |r_n-a|<\varepsilon\}$ is infinite for all $\varepsilon>0$.
By the main subsequences theorem, this ensures that there is a subsequence r_{n_k} that converges to a.

(c) Since r_n is unbounded above, the main subsequences theorem ensures that there is a subsequence that diverges to $+\infty$.

(4)

(a) Suppose s_n is a Cauchy sequence, according to our definition from class. Fix $\varepsilon>0$. Then there exists N s.t. $n,m>N$ ensures $|s_n-s_m|<\varepsilon$. In particular, if $n>m>N$, we have $|s_n-s_m|<\varepsilon$.

Now, suppose s_n is a Cauchy sequence, according to the new definition. Fix $\varepsilon>0$. Then $\exists$ N s.t. $k>l>N$ ensures $|s_k-s_l|<\varepsilon$. Suppose $n,m>N$. If $n=m$, then $|s_n-s_m|=0<\varepsilon$. If $n>m$, take $k=n$, $l=m$ to see $|s_n-s_m|<\varepsilon$. Lastly, if $n<m$, take $k=m$, $l=n$ to see $|s_n-s_m|<\varepsilon$.

ⓒ Since r_n is unbounded above, the main subsequences theorem ensures that there is a subsequence that diverges to $+\infty$.

④

ⓐ Suppose s_n is a Cauchy sequence, according to our definition from class. Fix $\varepsilon>0$. Then there exists N s.t. $n,m>N$ ensures $|s_n-s_m|<\varepsilon$. In particular, if $n>m>N$, we have $|s_n-s_m|<\varepsilon$.

Now, suppose s_n is a Cauchy sequence, according to the new definition. Fix $\varepsilon>0$. Then $\exists$ N s.t. $k>l>N$ ensures $|s_k-s_l|<\varepsilon$. Suppose $n,m>N$. If $n=m$, then $|s_n-s_m|=0<\varepsilon$. If $n>m$, take $k=n$, $l=m$ to see $|s_n-s_m|<\varepsilon$. Lastly, if $n<m$, take $k=m$, $l=n$ to see $|s_n-s_m|<\varepsilon$.

④ⓑ

$\sum_{k=1}^{\infty} a_k$ is convergent

$\Updownarrow$

$s_n = \sum_{k=1}^{n} a_k$ converges

$\Updownarrow$

s_n is Cauchy

$\Updownarrow$ ⓐ

$\forall \varepsilon > 0$, $\exists N \in \mathbb{R}$ so that $n > m > N$ ensures $|s_n - s_m| < \varepsilon$

$$\sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k = \sum_{k=m+1}^{n} a_k$$

$\Updownarrow$

$\forall \varepsilon > 0$, $\exists N \in \mathbb{R}$ so that $n > m > N$ ensures $\left|\sum_{k=m+1}^{n} a_k\right| < \varepsilon$

ⓒ Suppose $\sum_{k=1}^{\infty} a_k$ is convergent. WTS $\lim a_k = 0$. Fix $\varepsilon > 0$. By part ⓑ, $\exists N$ s.t. $n > m > N$ implies $\left|\sum_{k=m+1}^{n} a_k\right| < \varepsilon$. In particular, $\exists N$ s.t. $m > N$ and $n = m+1$ implies $|a_n| < \varepsilon$, so $|a_n - 0| < \varepsilon$. Thus $\lim a_k = 0$.

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⑥ ⓐ By definition $s_{n+1} = s_n + \frac{d_{n+1}}{10^{n+1}}$. Since $d_{n+1} \geq 0$, $s_{n+1} \geq s_n$.

ⓑ Taking $a = \frac{1}{10}$ in Q5(a) gives

$$1 + \frac{1}{10} + \frac{1}{10^2} + \dots + \frac{1}{10^n} = \frac{1 - (1/10)^{n+1}}{9/10}$$

$$\Leftrightarrow 9 + \frac{9}{10} + \frac{9}{10^2} + \dots + \frac{9}{10^n} = 10 - (1/10)^n$$

$$\Leftrightarrow \frac{9}{10} + \frac{9}{10^2} + \dots + \frac{9}{10^n} = 1 - \left(\frac{1}{10}\right)^n$$

ⓒ Since $s_n = K + \frac{d_1}{10} + \frac{d_2}{10^2} + \dots + \frac{d_n}{10^n}$ and $d_i \leq 9$ for all $i = 1, \dots, n$,

$s_n \leq K + \frac{9}{10} + \frac{9}{10^2} + \dots + \frac{9}{10^n} = K + 1 - \frac{1}{10^n} \leq K + 1$.

Therefore s_n is bounded above. Since $s_n \geq 0$, it is also bounded below, hence bounded.

(d) Let $s_n = .\overbrace{99\ldots9}^{n\text{ times}}$. Then $s_n = 1 - \frac{1}{10^{n+1}}$.

Since $\lim_{n\to\infty} \frac{1}{10^n} = 0$, $\lim_{n\to\infty} \frac{1}{10^{n+1}} = \lim_{n\to\infty} \frac{1}{10^n}\frac{1}{10} = 0$,

hence $\lim_{n\to\infty} 1 - \frac{1}{10^{n+1}} = 0$. Thus,

$.\overline{9} = \lim_{n\to\infty} s_n = 1$.

⑧ First, note that $\liminf \sigma_n \le \limsup \sigma_n$ by definition of liminf and limsup.

We now show $\liminf s_n \le \liminf \sigma_n$ by first proving the hint.

Note that if $n > M > N$

$$\sigma_n = \frac{1}{n}(s_1 + s_2 + \dots + s_n)$$
$$= \frac{1}{n}(s_1 + s_2 + \dots + s_N + s_{N+1} + \dots + s_M + \dots + s_n)$$

$s_i \ge 0$ for all i

$$\ge \frac{1}{n}(s_{N+1} + \dots + s_M + \dots + s_n)$$
$$\ge \frac{1}{n}(n-N)\inf\{s_n : n > N\}$$

since for $i > n$, $s_i > \inf\{s_n : n > N\}$ and there are $(n-N)$ elements in the sum

since $n > M$

$$= \left(1 - \frac{N}{n}\right)\inf\{s_n : n > N\}$$
$$\ge \left(1 - \frac{N}{M}\right)\inf\{s_n : n > N\}$$

Therefore $\left(1-\frac{N}{M}\right)\inf\{s_n : n > N\}$ is a lower bound for the set $\{\sigma_n : n > M\}$.

Hence $\underbrace{\inf\{\sigma_n : n > M\}}_{B_M} \ge \left(1-\frac{N}{M}\right)\underbrace{\inf\{s_n : n > N\}}_{b_N}$.

First suppose N is fixed. Since $B_M \ge \left(1-\frac{N}{M}\right) b_N$ for all $M > N$, sending $M \to +\infty$ gives $\liminf \sigma_n = \lim_{M\to\infty} B_M \ge b_N$.

Now, sending $N \to +\infty$ gives $\liminf \sigma_n \ge \lim_{N \to \infty} b_N = \liminf s_n$, which proves the first inequality.

Now we show $\limsup \sigma_n \le \limsup s_n$ by proving the other hint.

Note that if $n > M > N$,

$$\sigma_n = \frac{1}{n}(s_1 + s_2 + \dots + s_N + s_{N+1} + \dots + s_M + \dots + s_n)$$

$$= \frac{1}{n}(s_1 + s_2 + \dots + s_N) + \frac{1}{n}(s_{N+1} + \dots + s_M + \dots + s_n)$$

since for $i > N$ $s_i < \sup\{s_n : n > N\}$ and there are $(n-N)$ elements in the second sum

$$\le \frac{1}{n}(s_1 + s_2 + \dots + s_N) + \frac{1}{n}(n-N)\sup\{s_n : n > N\}$$

$$\le \frac{1}{n}(s_1 + s_2 + \dots + s_N) + \sup\{s_n : n > N\} \quad \leftarrow \frac{1}{n}(n-N) < 1$$

$$\underset{n>M}{\le} \frac{1}{M}(s_1 + s_2 + \dots + s_N) + \sup\{s_n : n > N\}$$

Thus $\sup\{\sigma_n : n > M\} \le \overbrace{\frac{1}{M}(s_1+s_2+\dots+s_N)}^{A_M} + \overbrace{\sup\{s_n : n > N\}}^{a_N}$.

Sending $M \to +\infty$ for fixed N gives,

$\limsup \sigma_n = \lim_{M\to\infty} A_M \le 0 + a_N$.

Then sending $N \to \infty$ gives

$\limsup \sigma_N \le \lim_{N\to\infty} a_N = \limsup s_n$,

which completes the proof.

(b) If $\lim s_n$ exists, then $\limsup s_n = \liminf s_n$. Hence, by part (a), $\limsup \sigma_n = \liminf \sigma_n$. Therefore $\lim \sigma_n$ exists.

(c) Consider $s_n = (-1)^{n+1}$, so $\lim s_n$ doesn't exist. Then

$$\sigma_n = \begin{cases} \frac{1}{n} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even,} \end{cases}$$

so $\lim \sigma_n = 0$.

10 (a) Claim: $S=\{0\}\cup\{\frac{1}{\ell}:\ell\in\mathbb{N}\}$

Since $s_{n_k}=\frac{1}{k}$ is a subsequence, $0\in S$. Since $s_{n_k}=\frac{1}{\ell}$ is a subsequence for all $\ell\in\mathbb{N}$, $\frac{1}{\ell}\in S$.

It remains to show no other real number or $\pm\infty$ belongs to S.

Neither $+\infty$ nor $-\infty$ belong to S, since the sequence is bounded.

Suppose $a\in S$ for some $a\in\mathbb{R}$. By the main subsequences theorem, it suffices to show $\exists\ \varepsilon_0>0$ so that $|a-s_n|\geq\varepsilon_0$ for all n.

$\boxed{\text{If } a>1}$, then $|a-s_n|\geq|a-1|=:\varepsilon_0\ \forall n$

$\boxed{\text{If } a<0}$, then $|a-s_n|>|a|=:\varepsilon_0\ \forall n$.

$\boxed{\text{If } \frac{1}{\ell}>a>\frac{1}{\ell+1} \text{ for some } \ell\in\mathbb{N}}$ then $|a-s_n|\geq\min\{|a-\frac{1}{\ell}|,|a-\frac{1}{\ell+1}|\}=:\varepsilon_0\ \forall n$

This completes the proof.

12 (a) If $\lim s_n = s$, then all subsequences of s_n also converge to s. Hence every subsequence s_{n_k} has a further subsequence $s_{n_{k_\ell}} = s_{n_k}$ that converges to s.

(b) Suppose $\lim s_n \neq s$. Then,
$\exists\, \varepsilon > 0$ s.t. $\forall N$, $\exists n > N$ s.t. $|s_n - s| \geq \varepsilon$

First, taking $N = 1$, we have $\exists\, n_1 > 1$ s.t. $|s_{n_1} - s| \geq \varepsilon$. Suppose we have chosen n_{k-1}. Taking $N = n_{k-1}$, we see that $\exists\, n_k > n_{k-1}$ s.t. $|s_{n_k} - s| \geq \varepsilon$.

Therefore there exists a subsequence s_{n_k} s.t. $|s_{n_k} - s| \geq \varepsilon$ $\forall k$. Since s_{n_k} is always at least distance ε from s, no further subsequence of s_{n_k} can converge to s.