# MATH 117: HOMEWORK 6

Due Thursday, May 14th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

### Question 1

Suppose $S \subseteq \mathbb{R}$ is a bounded set and $f$ is a bounded function on $S$. Define $M = \sup\{f(s) : s \in S\}$.

(a) Use the definition of the supremum to prove that there exists a sequence $s_n$ of elements in $S$ so that $\lim_{n\to+\infty} f(s_n) = M$.

(b) Now, use the result from part (a) to prove that there exists a *convergent* sequence $t_k$ of elements in $S$ so that $\lim_{k\to+\infty} f(t_k) = M$.

### Question 2*

Consider the set $S = \left\{\sqrt{5}r : r \in \mathbb{Q}\right\}$. You may assume that $\sqrt{5}$ is an irrational number.

(a) Prove that $S$ is dense in $\mathbb{R}$ by showing that, for all $a, b \in \mathbb{R}$ with $a < b$, there exists $s \in S$ satisfying $a < s < b$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that if $f(s) = \pi$ for all $s \in S$, then $f(x) = \pi$ for all $x \in \mathbb{R}$.

(c) Let $f$ and $g$ be continuous real-values functions defined on $\mathbb{R}$ such that $f(s) = g(s)$ for each $s \in S$. Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$. (Hint: define a new function and apply part (b).)

*As a consequence, you see that if two continuous functions on $\mathbb{R}$ are equal on a dense subset of $\mathbb{R}$, they must actually be equal everywhere.*

### Question 3*

Fix $a \in \mathbb{R}$ and consider the function

$$g_a(x) = \begin{cases} \frac{1}{x} & \text{for } x \neq 0, \\ a & \text{for } x = 0. \end{cases}$$

Prove that $g_a$ is not continuous.

### Question 4*

Suppose $f$ is a continuous function on $\mathbb{R}$ and $f(a)f(b) < 0$ for some $a, b \in \mathbb{R}$. Prove there exists $x$ between $a$ and $b$ such that $f(x) = 0$.

## Question 5*

We begin by defining what it means for a function to be convex:

**DEFINITION 1.** A function $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is *convex* if

$$f((1-\alpha)x + \alpha y) \leq (1-\alpha)f(x) + \alpha f(y) \ , \text{ for all } \alpha \in [0,1].$$

Prove that the following functions $f : \mathbb{R} \to \mathbb{R}$ are convex:

(a) $f(x) = -x$

(b) $f(x) = x^2$

(c) $f(x) = \max\{x, 0\}$

(d) $f(x) = |x|$

## Question 6*

The *extended real numbers* is the set $\overline{\mathbb{R}} := \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$. We may extend the binary operations of addition $+$ and multiplication $\cdot$ and the ordering $\leq$ from $\mathbb{R}$ to $\overline{\mathbb{R}}$ via the following rules:

(a) $\pm\infty + x = \pm\infty$ for all $x \in \mathbb{R}$.
$+\infty + (+\infty) = +\infty$ and $-\infty + (-\infty) = -\infty$.
$\pm\infty + (\mp\infty)$ is not defined.

(b) $\pm\infty \cdot x = \pm\infty$ for $x \in (0, +\infty]$ and $\pm\infty \cdot x = \mp\infty$ for $x \in [-\infty, 0)$.
$(\pm\infty) \cdot (\pm\infty) = +\infty$ and $(\pm\infty) \cdot (\mp\infty) = -\infty$.
$\pm\infty \cdot 0 = 0$

(c) $-\infty < x < +\infty$ for all $x \in \mathbb{R}$.

We now recall the definition of *convex* and *non-decreasing* functions.

**DEFINITION 2.** A function $f : \mathbb{R} \to \mathbb{R}$ is *non-decreasing* if

$$x \leq y \implies f(x) \leq f(y).$$

Using these definitions, we will now establish a few basic properties of convex functions.

(i) Suppose $f$ is a convex function satisfying $f(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$. Is $-f$ a convex function? Justify your answer.

(ii) Suppose $f$ and $g$ are convex functions and $c_1, c_2 \in \mathbb{R}$. Under what conditions on $c_1$ and $c_2$ is $c_1 f(x) + c_2 g(x)$ always a convex function? Prove that the conditions you find are sufficient and give an example to show that the result fails without those conditions.

(iii) Suppose $f$ is convex and $g$ is convex and increasing. Is $g \circ f$ convex? Is $f \circ g$ convex? Justify your answer with a proof or a counterexample.

## Question 7

We now recall the definition of an *affine* set.

**DEFINITION 3.** A set $C \subseteq \mathbb{R}^d$ is *affine* if the line through any two points in $C$ lies in $C$, that is, for all $x_0, x_1 \in C$,

$$(1-\alpha)x_0 + \alpha x_1 \in C, \quad \forall \alpha \in \mathbb{R}.$$

(a) Give an example of $C \subseteq \mathbb{R}^d$ that is affine but is not a subspace.

(b) Given $\{x_1, \dots, x_k\} \subseteq \mathbb{R}^d$, any point of the form

$$\sum_{i=1}^{k} \alpha_i x_i, \quad \text{such that } \sum_{i=1}^{k} \alpha_i = 1,$$

is an *affine combination* of $\{x_i\}_{i=1}^k$. If $C$ is affine, prove that $C$ is *closed under affine combinations*, that is, for any $k \in \mathbb{N}$, $\{x_i\}_{i=1}^k \subseteq C$, all affine combinations of $\{x_i\}_{i=1}^k$ are in $C$.

(c) If $C$ is an affine set and $x_0 \in C$, prove that

$$C - x_0 := \{x - x_0 : x \in C\}$$

is a subspace.

## Question 8*

We now recall the definition of a *convex* set and *convex combination.* (A convex combination can be thought of as a weighted average of the points.)

**DEFINITION 4.** A set $C \subseteq \mathbb{R}^d$ is *convex* if the line segment between any two points in $C$ lies in $C$, that is, for all $x_0, x_1 \in C$,

$$(1-\alpha)x_0 + \alpha x_1 \in C, \quad \forall \alpha \in [0,1].$$

**DEFINITION 5.** Given $\{x_1, \dots, x_k\} \subseteq \mathbb{R}^d$, any point of the form

$$\sum_{i=1}^{k} \alpha_i x_k, \quad \text{such that } \alpha_i \geq 0 \text{ and } \sum_{i=1}^{k} \alpha_i = 1,$$

is a *convex combination* of $\{x_i\}_{i=1}^k$

(a) Give an example of $C \subseteq \mathbb{R}^d$ that is convex but not affine.

(b) If $C$ is convex, prove that $C$ is *closed under convex combinations*, that is, for any $k \in \mathbb{N}$, $\{x_i\}_{i=1}^k \subseteq C$, all convex combinations of $\{x_i\}_{i=1}^k$ are in $C$.

(c) Conversely, if $C$ is *closed under convex combinations*, prove that $C$ is convex.

## Question 9*

Given a function $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, its *epigraph* is defined to be the set

$$\text{epi}(f) := \{(x,t) \in \mathbb{R}^d \times \mathbb{R} : t \geq f(x)\}.$$

Prove that $f$ is a convex function if and only if epi$(f)$ is a convex set.

## Question 10

(a) Given a function $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, its *effective domain* is the set $D(f) := \{x \in \mathbb{R}^d : f(x) \in \mathbb{R}\}$. Prove that, if $f$ is a convex function, $D(f)$ is a convex set.

(b) Given $C \subseteq \mathbb{R}^d$, the *convex indicator function* $\chi_C : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is given by

$$\chi_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{if } x \notin C. \end{cases}$$

Prove that $C$ is a convex set if and only if its indicator function $\chi_C$ is a convex function.

## Question 11

Given a function $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, prove that $f$ is convex if and only if

$$f(\lambda_1 x_1 + \cdots + \lambda_n x_n) \leq \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n)$$

for all $\{x_i\}_{i=1}^n \subseteq \mathbb{R}^d$ and $\{\lambda_i\}_{i=1}^n \subseteq [0, +\infty)$ satisfying $\sum_{i=1}^n \lambda_i = 1$.

## Question 12

A function $f : \mathbb{R}^d \to \mathbb{R}$ is *positively homogeneous* if, for every $\lambda > 0$, $f(\lambda x) = \lambda f(x)$. Prove that a positively homogeneous function $f : \mathbb{R}^d \to \mathbb{R}$ is convex if and only if

$$f(x + y) \leq f(x) + f(y), \quad \forall x, y \in \mathbb{R}^d.$$