Math 117: Homework 7 Solutions

Question 1

(a) Assume for the sake of contradiction that f is not bounded on S. Then $\not\exists\ M > 0$ s.t. $|f(x)| \leq M \forall x \in S$. Hence, $\forall n \in \mathbb{N}$, $\exists x_n \in S$ so that $|f(x_n)| > n$. Since $S \subseteq \mathbb{R}$ is bounded x_n is bounded. By Bolzano-Weierstrass, it has a convergence subsequence x_{n_k}. Since f is uniformly continuous, $f(x_{n_k})$ is also convergence. This contradicts the fact $f(x_{n_k})$ is unbounded. Thus, f must be bounded on S.

(b) No. Let $S = (0, 1)$ and define $f : S \to \mathbb{R}$ by

$$f(x) = \frac{1}{x}.$$

Then S is bounded and f is continuous on S, but f is not bounded on S.

Question 2

(a) Suppose first that f is lower semicontinuous at x. Let $x_n \to x$, and assume that $\lim_{n\to\infty} f(x_n)$ exists in $\overline{\mathbb{R}}$. Then

$$f(x) \leq \liminf_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(x_n).$$

Conversely, assume that for every sequence $x_n \to x$ for which $\lim_{n\to\infty} f(x_n)$ exists, we have

$$f(x) \leq \lim_{n\to\infty} f(x_n).$$

Let $x_n \to x$ be arbitrary. There exists a subsequence x_{n_k} such that

$$\lim_{k\to\infty} f(x_{n_k}) = \liminf_{n\to\infty} f(x_n).$$

Since $x_{n_k} \to x$,

$$f(x) \leq \lim_{k\to\infty} f(x_{n_k}) = \liminf_{n\to\infty} f(x_n),$$

so f is lower semicontinuous at x.

(b) Fix $x \in \mathbb{R}$, and define

$$m_x(\epsilon) := \inf\{f(y) : |y - x| \leq \epsilon\}.$$

Since $m_x(\epsilon)$ is increasing in ϵ, $\lim_{\epsilon\to 0^+} m_x(\epsilon)$ exists and

$$\liminf_{y\to x} f(y) = \lim_{\epsilon\to 0^+} m_x(\epsilon).$$

Suppose f is lower semicontinuous at x. By definition, we have

$$\liminf_{y\to x} f(y) = \lim_{\epsilon\to 0^+} m_x(\epsilon) \geq \lim_{\epsilon\to 0^+} f(x) = f(x).$$

Assume that $\liminf_{y\to x} f(y) < f(x)$. Choose $\alpha \in \mathbb{R}$ such that

$$\liminf_{y\to x} f(y) < \alpha < f(x).$$

Since $\lim_{n\to+\infty} m_x(1/n) = \liminf_{y\to x} f(y)$, for all sufficiently large n,

$$m_x(1/n) < \alpha.$$

Thus we may choose y_n such that

$$|y_n - x| \leq \frac{1}{n} \qquad \text{and} \qquad f(y_n) < \alpha.$$

Then $y_n \to x$, but

$$\liminf_{n\to\infty} f(y_n) \leq \alpha < f(x),$$

contradicting lower semicontinuity at x.

Conversely, suppose that $\liminf_{y\to x} f(y) = f(x)$. Let $x_n \to x$. For any $\epsilon > 0$, there exists N such that for all $n \geq N$,

$$|x_n - x| \leq \epsilon.$$

Therefore, for all $n \geq N$,

$$f(x_n) \geq \inf\{f(y) : |y - x| \leq \epsilon\} = m_x(\epsilon).$$

Hence

$$\liminf_{n\to\infty} f(x_n) \geq m_x(\epsilon).$$

Taking $\epsilon \to 0^+$, we obtain

$$\liminf_{n\to\infty} f(x_n) \geq \lim_{\epsilon\to 0^+} m_x(\epsilon) = \liminf_{y\to x} f(y) = f(x).$$

Thus f is lower semicontinuous at x.

Question 3

Suppose first that f is lower semicontinuous. Let $(x_n, t_n) \in \mathrm{epi}(f)$ satisfy

$$x_n \to x, \qquad t_n \to t.$$

Since $(x_n, t_n) \in \mathrm{epi}(f)$, we have $f(x_n) \leq t_n$ for every n. By lower semicontinuity,

$$f(x) \leq \liminf_{n\to\infty} f(x_n) \leq \lim_{n\to\infty} t_n = t.$$

Thus $(x, t) \in \mathrm{epi}(f)$.

Conversely, suppose $\mathrm{epi}(f)$ is sequentially closed in the stated sense. If f were not lower semicontinuous at some x, then there would exist a sequence $x_n \to x$ such that $f(x) > \liminf_{n\to+\infty} f(x_n)$. Choose a so that $f(x) > a > \liminf_{n\to+\infty} f(x_n)$. Then, since $\liminf_{n\to+\infty} f(x_n)$ is a subsequential limit, there exists x_{n_k} so that $\lim_{k\to+\infty} f(x_{n_k}) = \liminf_{n\to+\infty} f(x_n) < a$. There must be infinitely many terms in the sequence so that $f(x_{n_k}) < a$, otherwise we would have $\lim_{k\to+\infty} f(x_{n_k}) \geq a$. Let $x_{n_{k_l}}$ denote the subsequence satisfying $f(x_{n_{k_l}}) < a$.

Then $(x_{n_{k_l}}, a) \in \mathrm{epi}(f)$ for every l, and

$$(x_{n_{k_l}}, a) \to (x, a).$$

By the assumed closedness of the epigraph, $(x, a) \in \mathrm{epi}(f)$, so $f(x) \leq a$, which is a contradiction. Therefore f is lower semicontinuous.

Question 5

(a) Let $x_n \to x$. We want to show that

$$\liminf_{n\to\infty} f_*(x_n) \geq f_*(x).$$

If $f_*(x) = -\infty$, this is immediate. Otherwise, let $\alpha < f_*(x)$. Since

$$f_*(x) = \sup_{\epsilon>0} \inf\{f(y) : |x-y| < \epsilon\},$$

there exists $r > 0$ such that

$$\inf\{f(y) : |x-y| < r\} > \alpha.$$

Since $x_n \to x$, for all sufficiently large n,

$$|x_n - x| < \frac{r}{2}.$$

Then whenever $|y - x_n| < r/2$, we have

$$|y-x| \leq |y-x_n| + |x_n - x| < r.$$

Therefore

$$\inf\{f(y) : |y-x_n| < r/2\} \geq \inf\{f(y) : |y-x| < r\} > \alpha.$$

Since $f_*(x_n)$ is the supremum of these local infima,

$$f_*(x_n) > \alpha$$

for all sufficiently large n. Hence

$$\liminf_{n\to\infty} f_*(x_n) \geq \alpha.$$

Because this holds for every $\alpha < f_*(x)$, we get

$$\liminf_{n\to\infty} f_*(x_n) \geq f_*(x).$$

Thus f_* is lower semicontinuous.

(b) For every $\epsilon > 0$, since $x \in \{y : |x-y| < \epsilon\}$, we have

$$\inf\{f(y) : |x-y| < \epsilon\} \leq f(x).$$

Taking the limit as $\epsilon \to 0^+$, we obtain

$$f_*(x) \leq f(x).$$

(c) First, let $x_n \to x$. Since $f_* \leq f$, we have

$$f_*(x_n) \leq f(x_n)$$

for every n. Therefore

$$\liminf_{n\to\infty} f_*(x_n) \leq \liminf_{n\to\infty} f(x_n).$$

By part (a), f_* is lower semicontinuous, so

$$f_*(x) \le \liminf_{n\to\infty} f_*(x_n).$$

Thus

$$f_*(x) \le \liminf_{n\to\infty} f(x_n)$$

for every sequence $x_n \to x$. Taking the infimum over all such sequences gives

$$f_*(x) \le \inf\left\{\liminf_{n\to\infty} f(x_n) : x_n \to x\right\}.$$

For the reverse inequality, for each $n \in \mathbb{N}$, choose x_n such that

$$|x_n - x| < \frac{1}{n}$$

and

$$f(x_n) \le \inf\{f(y) : |x-y| < 1/n\} + \frac{1}{n}.$$

Then $x_n \to x$. Hence

$$\liminf_{n\to\infty} f(x_n) \le \lim_{n\to\infty}\left(\inf\{f(y) : |x-y| < 1/n\} + \frac{1}{n}\right) = f_*(x).$$

Therefore

$$\inf\left\{\liminf_{n\to\infty} f(x_n) : x_n \to x\right\} \le f_*(x).$$

Combining both inequalities,

$$f_*(x) = \inf\left\{\liminf_{n\to\infty} f(x_n) : x_n \to x\right\}.$$

Question 6

(a) Assume $\mathcal{A}$ is nonempty. Let $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$. For $\lambda = 0$ or $\lambda = 1$, the convexity inequality is immediate, so suppose $0 < \lambda < 1$. Since each f_γ is convex,

$$f_\gamma(\lambda x + (1-\lambda)y) \le \lambda f_\gamma(x) + (1-\lambda) f_\gamma(y)$$

for every $\gamma \in \mathcal{A}$. Taking the supremum over γ gives

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= \sup_{\gamma\in\mathcal{A}} f_\gamma(\lambda x + (1-\lambda)y) \\ &\le \sup_{\gamma\in\mathcal{A}} \left(\lambda f_\gamma(x) + (1-\lambda) f_\gamma(y)\right) \\ &\le \lambda \sup_{\gamma\in\mathcal{A}} f_\gamma(x) + (1-\lambda) \sup_{\gamma\in\mathcal{A}} f_\gamma(y) \\ &= \lambda f(x) + (1-\lambda) f(y). \end{aligned}$$

Thus f is convex.

(b) Fix $x \in \mathbb{R}$, and let $x_n \to x$. We want to show that

$$\liminf_{n\to\infty} f(x_n) \geq f(x).$$

For every $\gamma \in \mathcal{A}$, we have

$$f(x_n) = \sup_{\alpha\in\mathcal{A}} f_\alpha(x_n) \geq f_\gamma(x_n).$$

Therefore,

$$\liminf_{n\to\infty} f(x_n) \geq \liminf_{n\to\infty} f_\gamma(x_n).$$

Since f_γ is lower semicontinuous and $x_n \to x$, the sequential definition gives

$$\liminf_{n\to\infty} f_\gamma(x_n) \geq f_\gamma(x).$$

Hence, for every $\gamma \in \mathcal{A}$,

$$\liminf_{n\to\infty} f(x_n) \geq f_\gamma(x).$$

Taking the supremum over all $\gamma \in \mathcal{A}$, we obtain

$$\liminf_{n\to\infty} f(x_n) \geq \sup_{\gamma\in\mathcal{A}} f_\gamma(x) = f(x).$$